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Discussion on Longitude Differences of Catalogues in View of Determining Ephemeris Time

Shinko Aoki
Tokyo Astronomical Observatory, University of Tokyo, Tokyo, Japan
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The systematic differences among star catalogues are rediscussed in view of determining the correction to be applied to UT in order to give ET if the *Improved Lunar Ephemeris* (or the moon's ephemeris in the *Astronomical Ephemeris* for 1960 on) is used. Discussions concerning the above circumstance in the prediction and the reduction of solar eclipses are given. It is noted that the conventional ΔT (ΔT_0) is preferable for use in the prediction of solar eclipses. A method of determining ET from the reduction of solar eclipses is discussed.

1. INTRODUCTION

Ephemeris Time is practically determined from the comparison of observed longitudes of the moon with the *Improved Lunar Ephemeris* (see pages viii, xii, for example, of the Introduction). The 10th General Assembly of the I.A.U., held at Moscow in 1958, has, however, adopted the recommendation that Ephemeris Time is fundamentally defined through the mean longitudes of the sun. Therefore some connection must be established between two systems, the mean longitudes of the moon, and of the sun. In fact, such a connection has already been given in Spencer Jones' work (1932, 1939), and it was adopted for the revision of the lunar tables (*Improved Lunar Ephemeris* 1954). It seems, however, that there is some confusion when comparing the equinox correction of the star catalogues and the moon's observation with the B term in Spencer Jones' papers (Brouwer 1952; Gondolatsch 1953; Markowitz 1959; Murray 1959)

I have published a note with regard to this point (Aoki 1960). One conclusion of the note was that Spencer Jones' revision (1932) of Newcomb's reduction of occultations (1912) is practically referred to the equinox of Hedrick's *Zodiacal Star Catalogue* (1905). Making the same assumption as Hedrick (p. 414), it was supposed in that note that the equinox of Hedrick's catalogue was the same as that of Newcomb's (1905).

At the present time, however, reductions of occultation observations are usually referred to Robertson's *Zodiacal Star Catalogue* (1940). Hence if we wish to derive B from the observed mean longitudes of the moon from occultation observations and its value is to be consistent with one from old observations, it should be reduced to the same star system as the old one by making allowance for the catalogue difference, "Hed–Rob." The same procedure is needed in the case of determining of ET from comparison of the moon's observation with the *Improved Lunar Ephemeris*. I have tried such a procedure in the previous note. However, in that discussion only the difference of the constant terms of right ascension was introduced, neglecting the gradual change of that difference with time. That change will become a considerable amount after a long time interval. Therefore the difference of the motion of the equinoxes of the catalogues should be taken into account as well as that of the equinox corrections. The effect of systematic difference of declination upon the mean longitude, though it may be slight, should also be taken into account.

In the present paper the differences among catalogues are re-examined and a definitive longitude difference which should be added to the observed mean longitudes of the moon in determining ET is proposed.

Also proposed is the conventional value of ΔT, because it is inconvenient to use the different equinoxes of reference for the moon and the sun when predicting solar eclipses, for example. The present method is similar to the one by Murray and Sadler (unpublished), but starts from a different point of view. Furthermore, the complicated circumstances in the reduction of solar

TABLE I. Systematic differences between catalogues.

	Hed−Rob	N30−FK3	Rob−FK3
$\Delta\alpha_0$	$0^{s}.053_5 - 0^{s}.003_0T$ $\pm 0''.000_6 \quad \pm 0''.000_1$	$0.010_1 + 0.003_0T'$ $\pm 0.000_7 \quad \pm 0.002_3$	$0.000_8 + 0.000_4T'$ $\pm 0.000_2 \quad \pm 0.000_2$
$\Delta\delta_c$	$-0''.03_7 + 0''.01_9T$ $\pm 0.00_8 \quad \pm 0.04_3$	$-0.05_1 - 0.14_4T'$ $\pm 0.00_4 \quad \pm 0.00_9$	$-0.00_8 - 0.08_3T'$ $\pm 0.00_2 \quad \pm 0.00_9$
ΔL	$0''.72_8 - 0''.03_7T$ $\pm 0.00_8 \quad \pm 0.00_8$	$0.12_2 + 0.01_2T$ $\pm 0.01_9 \quad \pm 0.03_1$	$0.01_4 - 0.01_1T$ $\pm 0.00_5 \quad \pm 0.00_4$

eclipses are mentioned. In Sec. 6 the numerical values of the ΔT's obtained in Japan are given for the years 1954–58.

2. LONGITUDE DIFFERENCES

Differentiating equations for the conversion of right ascension and declination into longitude and latitude, we can easily obtain the following equations:

$$\Delta\beta = \frac{1}{\cos\beta}\{(\cos\epsilon\cos\delta + \sin\epsilon\sin\delta\sin\alpha)\Delta\delta - \sin\epsilon\cos\delta\cos\alpha\Delta\alpha\}, \quad (1)$$

$$\Delta\lambda = \frac{\sin\beta\sin\lambda}{\cos\beta\cos\delta\cos\alpha}\{(\cos\epsilon\cos\delta + \sin\epsilon\sin\delta\sin\alpha)\Delta\delta - \sin\epsilon\cos\delta\cos\alpha\Delta\alpha\} + \frac{\sin\epsilon\cos\delta - \cos\epsilon\sin\delta\sin\alpha}{\cos\delta\cos\alpha}\Delta\delta + \cos\epsilon\Delta\alpha. \quad (2)$$

If we put $\beta \approx 0$ in these equations, we have

$$\Delta\lambda \approx \sin\epsilon\cos\alpha\Delta\delta + \cos\epsilon\Delta\alpha, \quad (3)$$

from which the mean longitude difference ΔL is given as follows:

$$\Delta L = \Delta\lambda_0 = \tfrac{1}{2}\sin\epsilon\Delta\delta_c + \cos\epsilon\Delta\alpha_0, \quad (4)$$

where it is supposed ΔA, like $\Delta\lambda$, $\Delta\alpha$, and $\Delta\delta$ is, in general, expressed in the following form:

$$\Delta A = \Delta A_0 + \Delta A_c\cos\alpha + \Delta A_s\sin\alpha, \quad (5)$$

because only the constant term and sine and cosine terms in terms of right ascension are needed in the remainder of this paper. $\Delta\alpha_0$ and $\Delta\delta_c$ are obtained from the comparison of the catalogues as listed in Table I, in which T denotes centuries elapsed since 1900.0 and T' denotes those since 1950.0.

From Eq. (4) we can obtain the mean longitude differences as listed in the last row of the Table I. And, although it is not directly needed in the rest of this paper, we can obtain from these values the correction which should be applied to the observations of the moon's longitudes, which have been referred to FK3, in order to be consistent with the equinox of the *Improved Lunar Ephemeris*,

$$\Delta L_{\mathrm{Hed-FK3}} = 0''.742 - 0''.048T \pm 0.009 \pm 0.009, \quad (6)$$

which is in close agreement with the value Newcomb −FK3 obtained by Murray (1960) in another way. This shows another verification of the near-coincidence of the equinox of Hedrick's catalogue with that of Newcomb's, and the value, Newcomb−FK3, given in the previous note, is uncertain.

Now, we define ΔT_0 in the following way:

$$L_{\text{☾ Rob}} = I.L.E.(U + \Delta T_0); \quad (7)$$

and from the above argument we should take ΔT, the correction to be applied to UT in order to give ET, as follows:

$$L_{\text{☾ Hed}} = I.L.E.(U + \Delta T), \quad (8)$$

where $L_{\text{☾ }X}$ denotes the moon's observed longitude referred to the star catalogue X, and $I.L.E.(T)$, in general, denotes the mean longitude given in the *Improved Lunar Eephemeris* at the instant T, and U denotes Universal Time. Then

$$\delta T \equiv \Delta T - \Delta T_0 = 1^{s}.82144(L_{\mathrm{Hed}} - L_{\mathrm{Rob}}) = 1^{s}.33 - 0^{s}.07T, \quad (9)$$
$$\pm 0.02 \pm 0.02.$$

According to the author's opinion, even at present the most reliable materials for the moon are occultation observations relating to the star catalogue. In this case, however, the deviations of the catalogue from the ideal system are directly transferred to the residuals. These should be corrected by finding the systematic corrections which may consist of corrections to the orbital elements as well as corrections of the star catalogues used. Of course, an occultation result gives only the combination of residuals in right ascension and declination, so that the degree of accuracy of separation of the unknowns is not very high; therefore the results should be compared with those of meridian observations, although a single observation of the latter is of lower accuracy.

In this way, as in the works of Spencer Jones, even at present in order to find Ephemeris Time we should take up primarily the occulation observations, the accuracy of which seems to reach $0''.1$, for an annual mean, when we use the photoelectric equipment (Osawa 1950; Hirose 1956, 1959; Hirose *et al.* 1958) with regard to the limb corrections (Aoki and Endô 1960a).

3. PROPOSAL FOR A REVISION OF THE IMPROVED LUNAR EPHEMERIS

The argument of Sec. 2 leads logically to the need for a revision of the *Improved Lunar Ephemeris*, if the absolute equinox is different from that of Hedrick's catalogue, or Newcomb's *Fundamental Star Catalogue.* However, it is not an easy way to find an absolute equinox, and to find it a great amount of observational materials of the sun and the moon and planets should be re-examined. This has been recently worked out by H. R. Morgan, and he has given data on the N30 star system (1952).

The equinox of N30 is, of course, consistent neither with that of Newcomb's nor with that of FK3. If we assume the equinox of N30 as an ideal one, then we can express the correction ΔL, which should be added to the mean longitudes of the moon given in the *Improved Lunar Ephemeris* in order to make it consistent with the solar longitudes given by Spencer Jones' discussions, as follows:

$$L_{\mathrm{N}30}=I.L.E.(U+\Delta T)+\Delta L, \qquad (10)$$

$$\begin{aligned}\Delta L&=L_{\mathrm{N}30}-L_{\mathrm{Hed}},\\ &=-0''.621+0''.060T, \qquad (11)\\ &\qquad \pm 0.021 \pm 0.032.\end{aligned}$$

As an alternative, we may express ΔT_1 in the following form:

$$L_{\mathrm{N}30}=I.L.E.(U+\Delta T_1), \qquad (12)$$

with

$$\delta T_1 \equiv \Delta T_1-\Delta T_0=0^s.20+0^s.04T. \qquad (13)$$

We might mention in this connection that Markowitz (1959) considered the *Improved Lunar Ephemeris* to be referred to an ideal equinox, without regard to the above-mentioned circumstance and that he gave ΔT in the following expression:

$$L_{\mathrm{N}30}=I.L.E.(U+\Delta T),$$

and further that he corrected the values obtained by Brouwer by assuming that as right ascension differences Newcomb $-\mathrm{N}30=0^s.040$, $\mathrm{FK3}-\mathrm{N}30=-0^s.009$. From the above-mentioned argument, the values ΔT given by Markowitz, as denoted above by ΔT_1, should not be considered as corrections to be applied to UT in order to give ET, but satisfy the relation

$$\Delta T_1-\Delta T=-1^s.13+0^s.11T. \qquad (14)$$

4. PREDICTION OF SOLAR ECLIPSES AND THE CONVENTIONAL ΔT

If the revision of the *Improved Lunar Ephemeris* is not practical, then we are in a position to use it as it is. In this case, as far as the purpose of determining Ephemeris Time is concerned, the expressions (7) and (9) give the value of ΔT and the ET, but it is not very convenient, since the referred equinox of the moon is different from that of the sun. This circumstance causes some confusion in the case of the prediction and the reduction of solar eclipses, because the ephemeris position derived from Newcomb's solar table with Ephemeris Time as the argument of time is considered to be referred to the absolute equinox; on the other hand, the moon's ephemeris position derived from the *Improved Lunar Ephemeris* with Ephemeris Time as the argument of time is referred to Hedrick's catalogue (or Newcomb's F.S.C.). As far as prediction is concerned, the conventional ΔT is to be rather recommended, which Murray and Sadler have proposed from another point of view (unpublished). In the author's opinion, the above-mentioned ΔT_0, which is nearly the same as that of Murray and Sadler, may serve this purpose. If the equinox of N30 is again an ideal one, corrections for the errors made using ΔT_0 as ΔT are as follows:

To be added to the sun's mean longitude:

$$0''.055-0''.003T, \quad (=n_\odot \delta T).$$

To be added to that of the moon:

$$0''.108+0''.023T, \quad (=L_{\mathrm{N}30}-L_{\mathrm{Rob}}).$$

The difference, $-0''.053-0''.026T$, between them is within the accuracy of extrapolation to several years forward and of determination of the single observation. The effect on the position of the observer is negligible because the position is connected with the FK3 equinox through UT, and the influence of the difference between FK3 and N30 would be reduced by a factor 1/60.

5. REDUCTION OF SOLAR ECLIPSES

In the case of solar eclipses, the observation gives the sun's position referred to the moon's position, whose position referred to the star catalogue gives ET; therefore the sun's position referred to the star catalogue is given in terms of Ephemeris Time. In this way, if the observed position is different from that of Newcomb's tables, it would give the correction to the solar tables, assuming that the *Improved Lunar Ephemeris* correctly gives the position of the moon in terms of ET except for the question of the equinox. This point of view is clearly based on that of Recommendation No. 7 of Commission 4 at the 8th General Assembly of the I.A.U. As stated above, however, it was recommended at the 10th General Assembly that the mean longitudes of the sun should not be altered from those of Newcomb's.

Therefore such a discrepancy would again require a further revision of the *Improved Lunar Ephemeris*. This gives another direct determination of ET which is concerned with the connection of the mean longitude of the sun and the moon given by Spencer Jones.

In order to make clear the above discussion we write as follows:

$$L_{\leftmoon\,\mathrm{N30}}=I.L.E(U+\Delta T')+\Delta L+\delta L, \tag{14}$$

$$\begin{aligned} L_{\odot\,\mathrm{N30}}-L_{\leftmoon\,\mathrm{N30}}=&\,Solar\ Table(U+\Delta T')\\ &-I.L.E.(U+\Delta T')-\Delta L-\delta L, \end{aligned} \tag{15}$$

where $\Delta T'$ is the revised correction, $\Delta L=L_{\mathrm{N30}}-L_{\mathrm{Hed}}$, and δL is the further correction to the *Improved Lunar Ephemeris* as stated above. Let the observations, for brevity, be zero in the left-hand side of Eq. (15) at the instant U; we may write

$$C-O=Solar\ Table(U+\Delta T)-I.L.E.(U+\Delta T)-\Delta L \tag{16}$$

($O=0$ at instant U from the observations). The corrections to the other orbital elements of the sun and the moon might be added to the solar and the lunar tables except for the mean longitudes. After applying these appropriate corrections and the other corrections such as the limb effect, we may write $C'-O'$ instead of $C-O$, and denote it by (16′); then we obtain from (10), (14), (15), (16′), the following conditions:

$$\left.\begin{aligned} (n_{\leftmoon}-n_{\odot})(\Delta T'-\Delta T)+\delta L=C'-O',\\ \Delta T'-\Delta T+1^{\mathrm{s}}.821\delta L=0, \end{aligned}\right\} \tag{17}$$

from which we obtain

$$\left.\begin{aligned} \Delta T'-\Delta T=-1^{\mathrm{s}}.8214\frac{n_{\leftmoon}}{n_{\odot}}(C'-O')=-24^{\mathrm{s}}.3(C'-O')\\ \delta L=\frac{n_{\leftmoon}}{n_{\odot}}(C'-O')=13.4(C'-O'). \end{aligned}\right\} \tag{18}$$

As an alternative, if we consider that the further revision of the *Improved Lunar Ephemeris* is not so certain, we might take Eqs. (15) and (16′) making δL zero without regard to Eq. (14). This would be a third way of finding ET, and is given by the following expression:

$$\begin{aligned} \Delta T'-\Delta T=&+1^{\mathrm{s}}.8214\frac{n_{\leftmoon}}{n_{\leftmoon}-n_{\odot}}(C'-O')\\ =&+1^{\mathrm{s}}.967(C'-O'). \end{aligned} \tag{19}$$

6. NUMERICAL RESULTS OF ΔT'S

We have computed ΔT from the occultations observed by members of the staff of the Tokyo Astronomical Observatory (Hirose *et al.* 1959) with the photoelectric equipments during 1954–1958 (Aoki and Endô 1960a). The results are given in the 3rd column of Table II (listed as $\Delta T_{\mathrm{Japan}}$). These are slightly different from those of the original paper by the amount $\delta T-1^{\mathrm{s}}.09$, because the original values are referred to Brouwer's paper. It is to be noted that results obtained directly from observations in Japan have shown a systematic discrepancy from world-wide ones because of the deviations of the vertical in Japan (Hirose 1948). Although this amount is not definitely determined, I have taken $\Delta\lambda=20''$ (west longitude +), $\Delta\varphi=10''$ as round numbers, where $\Delta\lambda$ and $\Delta\varphi$ are the corrections to the geocentric longitude and latitude of the observer. The values of ΔT (Aoki and Endô 1960b) so corrected are given in the 4th column of Table II.

TABLE II. Annual values of ΔT derived from photoelectrically observed occultations at the Tokyo Astronomical Observatory.

Epoch	$\Delta T_{0\ \mathrm{Japan}}$	$\Delta T_{\mathrm{Japan}}$	ΔT
1054.5	$30^{\mathrm{s}}.11\pm0^{\mathrm{s}}.12$	$31^{\mathrm{s}}.40$	$31^{\mathrm{s}}.08$
1955.5	$30.51+0.14$	31.80	31.42
1956.5	30.44 ± 0.14	31.73	31.35
1957.5	31.50 ± 0.18	32.79	32.33
1958.5	31.38 ± 0.12	32.67	32.24

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Variation of Parameters for Elliptic and Near Circular Orbits

SAMUEL PINES
Republic Aviation Corporation, Farmingdale, New York

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The classical parameters used in the variation of constants for the special perturbation techniques in celestial mechanics present basic difficulties for orbits of small eccentricity and low inclination. In addition these methods require the solution of a differential equation for the integration of the mean anomaly using some form of Kepler's equation. A method is presented herein which avoids the ambiguities for near circular and low inclination orbits, and avoids the need of an additional equation for the integration of the mean motion. The method uses as parameters a set of initial position and velocity vectors in the osculating orbit plane.

INTRODUCTION

THE variation of parameters has always appealed to dynamical astronomers from the point of view of simplicity of derivation and accuracy of numerical computation. The method is a direct application of the Lagrange variation of constants to the nonlinear differential equations of a body with negligible mass in a multibody gravitational field under the influence of small perturbation forces.

The intent is to replace the determination of the rapidly varying instantaneous position and velocity of the body by the evaluation of the set of more slowly varying elements of an osculating two-body problem conic section. In some methods (Planetary Coordinates, 1960–1980, p. 156) the parameters used are vectors parallel and perpendicular to the perigee position. As the osculating ellipse approaches a circular orbit, the perigee vector assumes an ambiguous location, since there is no orbital point closest to the main attracting mass. Other methods (Planetary Coordinates, 1960–1980, p. 150) use the ascending node as a parameter. This parameter also attains ambiguous values for near equatorial orbits.

Almost all variational methods use the eccentric anomaly, the true anomaly, or the mean anomaly, referred to the perigee position to compute the instantaneous position. Since the perturbations tend to slowly change the perigee position, an additional variational equation arises.

By choosing a set of parameters which describes the two-body conic section without specific reference to the perigee position, the orbital inclination, or the ascending node, the difficulties described above may be eliminated, and an additional perturbation equation for the rapidly changing mean anomaly may be avoided. The parameters used in this report are the initial position and velocity vectors in the osculating plane, whose positions are related to the instantaneous position and velocity through Kepler's equation.

NOTATION

$\mathbf{R}$	Position vector in three-dimensional Cartesian coordinates
$\dot{\mathbf{R}}$	Velocity vector
$\mathbf{F}$	Perturbation acceleration vector
t	Time
$\mathbf{R}_0$	Initial position
$\dot{\mathbf{R}}_0$	Initial velocity
f, g	Functions related to the two-body problem
a	Semimajor axis
r_0	Scalar magnitude of $\mathbf{R}_0$
θ	Anomaly defined by the difference between the instantaneous and the initial eccentric anomaly
n	Mean angular velocity
T	Perturbation time parameter
v_0	Scalar magnitude of $\dot{\mathbf{R}}_0$
μ	The product of the universal gravitational constant and the mass of the reference body.

PERTURBATION EQUATIONS OF MOTION

The equations of motion of a body in a gravitational field subject to the acceleration of a homogeneous spherical mass in addition to smaller perturbation accelerations are given by

$$\ddot{\mathbf{R}} = -(\mu/r^3)\mathbf{R} + \mathbf{F}. \tag{1}$$

The first term on the right-hand side is the gravitational attraction of the dominant single mass. The second term represents the net perturbation acceleration. A solution of Eq. (1) in the absence of perturbations is given by

$$\begin{aligned} \mathbf{R} &= f(t,\mathbf{R}_0,\dot{\mathbf{R}}_0)\mathbf{R}_0 + g(t,\mathbf{R}_0,\dot{\mathbf{R}}_0)\dot{\mathbf{R}}_0, \\ \dot{\mathbf{R}} &= f_t(t,\mathbf{R}_0,\dot{\mathbf{R}}_0)\mathbf{R}_0 + g_t(t,\mathbf{R}_0,\dot{\mathbf{R}}_0)\dot{\mathbf{R}}_0. \end{aligned} \tag{2}$$

The vectors $\mathbf{R}_0$, $\dot{\mathbf{R}}_0$ are the initial position and velocity vectors, and the scalar functions f, g, f_t, g_t, are given by

$$\begin{aligned} f &= (a/r_0)(\cos\theta - 1) + 1, \\ g &= t - [(\theta - \sin\theta)/n], \\ f_t &= -(a^2 n/r r_0)\sin\theta, \\ g_t &= (a/r)(\cos\theta - 1) + 1. \end{aligned} \tag{3}$$

Kepler's equation relating the time with the angular variable θ is

$$nt = \theta - \sin\theta\left(1 - \frac{r_0}{a}\right) - (\cos\theta - 1)\frac{\mathbf{R}_0 \cdot \dot{\mathbf{R}}_0}{a^2 n}. \tag{4}$$

Equations (2) may be inverted to express the vectors $\mathbf{R}_0$ and $\dot{\mathbf{R}}_0$ as functions of the instantaneous position and velocity $\mathbf{R}$ and $\dot{\mathbf{R}}$. To accomplish this it is necessary only to replace t by $-t$, θ by $-\theta$, $\mathbf{R}$ by $\mathbf{R}_0$ and $\dot{\mathbf{R}}$ by $\dot{\mathbf{R}}_0$. Equations (2) become

$$\mathbf{R}_0=g_t\mathbf{R}-g\dot{\mathbf{R}},\quad \dot{\mathbf{R}}_0=-f_t\mathbf{R}+f\dot{\mathbf{R}}. \tag{5}$$

It is desired to obtain a solution of Eq. (1) in the form of Eqs. (2). The vectors $\mathbf{R}_0$ and $\dot{\mathbf{R}}_0$ may now be considered as slowly varying parameters which define the osculating plane of the orbit. The three second-order differential equations given by Eq. (1) may be replaced by six first-order perturbation differential equations describing the time variation of $\mathbf{R}_0$, $\dot{\mathbf{R}}_0$ due to the perturbation accelerations. In order to distinguish the perturbation time derivatives from the ordinary time derivatives, the perturbation variations shall be taken with respect to the time variable T. For example, the perturbation derivatives of the instantaneous position and velocity vectors are

$$\mathbf{R}_T=0,\quad \dot{\mathbf{R}}_T=\mathbf{F}. \tag{6}$$

The perturbation differential equation for the parameters $\mathbf{R}_0$ and $\mathbf{R}_0$ are given by differentiating Eqs. (5) with respect to T

$$\begin{aligned}\frac{d}{dt}\mathbf{R}_0&=g_{tT}\mathbf{R}-g_T\dot{\mathbf{R}}-g\mathbf{F},\\ \frac{d}{dt}\dot{\mathbf{R}}_0&=-f_{tT}\mathbf{R}+f_T\dot{\mathbf{R}}+f\mathbf{F}.\end{aligned} \tag{7}$$

The following relations are easily derived:

$$\begin{aligned}f_T&=(\cos\theta-1)\left(\frac{a}{r_0}\right)_T-\frac{a}{r_0}\sin\theta\,\theta_T,\\ g_T&=-(\theta-\sin\theta)\left(\frac{1}{n}\right)_T+\frac{\cos\theta-1}{n}\theta_T,\\ f_{tT}&=-\frac{\sin\theta}{r}\left(\frac{a^2n}{r_0}\right)_T-\frac{a^2n}{rr_0}\cos\theta\,\theta_T,\\ g_{tT}&=\frac{\cos\theta-1}{r}a_T-\frac{a}{r}\sin\theta\,\theta_T.\end{aligned} \tag{8}$$

The perturbation derivatives of the variables a, θ, n, and a/r_0 with respect to T are

$$\begin{aligned}a_T&=\frac{2a^2}{\mu}\dot{\mathbf{R}}\cdot\mathbf{F},\\ \left(\frac{1}{n}\right)_T&=\frac{3a}{n\mu}\dot{\mathbf{R}}\cdot\mathbf{F},\\ \theta_T&=\frac{1-\cos\theta}{r_0an}\mathbf{R}\cdot\mathbf{F}+\frac{g-3t+(r/an)\sin\theta}{r_0an}\dot{\mathbf{R}}\cdot\mathbf{F},\\ \left(\frac{a}{r_0}\right)_T&=\frac{a}{r_0^3}g\mathbf{R}\cdot\mathbf{F}-\frac{a}{r_0^3}\left\{\frac{\mathbf{R}_0\cdot\dot{\mathbf{R}}_0}{a^2n^2}\left(3t-g-\frac{r}{an}\sin\theta\right)\right.\\ &\qquad\left.+\frac{r_0}{an}\left(\frac{\mathbf{R}\cdot\dot{\mathbf{R}}}{a^2n^2}\sin\theta-\frac{2r}{an}\cos\theta\right)\right\}\dot{\mathbf{R}}\cdot\mathbf{F},\\ \left(\frac{a^2n}{r_0}\right)_T&=-\frac{1}{r_0n}\dot{\mathbf{R}}\cdot\mathbf{F}+an\left(\frac{a}{r_0}\right)_T.\end{aligned} \tag{9}$$

It is seen that the derivatives of the parameters $\mathbf{R}_0$ and $\dot{\mathbf{R}}_0$ as given by Eqs. (7) are of the magnitude of the perturbation acceleration $\mathbf{F}$. If this perturbation acceleration is small, the parameters will have the desired slowly varying character.

INTEGRATION OF THE EQUATIONS

A brief discussion of the integration of the perturbation differential equations given by Eqs. (7) will prove useful. In general, the vector $\mathbf{F}$ may be considered to be a function of the position $\mathbf{R}$, the velocity $\dot{\mathbf{R}}$, and the time t. The derivatives of the parameters $\mathbf{R}_0$ and $\dot{\mathbf{R}}_0$ may be evaluated from Eqs. (7). Since these are first-order nonlinear ordinary differential equations, some numerical integration scheme must be used to integrate them to obtain the values of $\mathbf{R}_0$ and $\dot{\mathbf{R}}_0$ as functions of time. The semimajor axis a, the mean motion n, and the scalar distance r_0, may be evaluated from the following:

$$\begin{aligned}1/a&=(2/r_0)-(v_0^2/\mu),\\ n&=\mu^{\frac{1}{2}}a^{-\frac{3}{2}},\\ r_0&=(\mathbf{R}_0\cdot\mathbf{R}_0)^{\frac{1}{2}},\\ v_0&=(\dot{\mathbf{R}}_0\cdot\dot{\mathbf{R}}_0)^{\frac{1}{2}}.\end{aligned} \tag{10}$$

The value of the anomaly θ for the new time t may be evaluated from Eq. (4) by Newton's method or by some iteration procedure. The values of $\mathbf{R}$ and $\dot{\mathbf{R}}$ may be obtained from Eqs. (2). The perturbation acceleration $\mathbf{F}$ may now be computed for the new time t and the process continued.

APPENDIX

Derivation of Perturbation Derivatives

The procedure for obtaining the perturbation derivative of a variable consists of the following: (1) Express the variable in terms of the instantaneous position $\mathbf{R}$ and the instantaneous velocity $\dot{\mathbf{R}}$. (2) Since the perturbation derivatives of these two variables are known, the differentiation of the desired variable may be carried out in a routine fashion.

The perturbation derivative of the anomaly θ is obtained below. From Kepler's law

$$nt=\theta-\sin\theta\left(1-\frac{r}{a}\right)+(\cos\theta-1)\frac{\mathbf{R}\cdot\dot{\mathbf{R}}}{a^2n}. \tag{A.1}$$

Differentiating with respect to T

$$n_T t=\left[1-\cos\theta\left(1-\frac{r}{a}\right)-\sin\theta\frac{\mathbf{R}\cdot\dot{\mathbf{R}}}{a^2n}\right]\theta_T$$
$$+r\sin\theta\left(\frac{1}{a}\right)_T+(\cos\theta-1)\left(\frac{\mathbf{R}\cdot\dot{\mathbf{R}}}{a^2n}\right)_T. \qquad \text{(A.2)}$$

The coefficient of θ_T may be recognized to be

$$1-\cos\theta\left(1-\frac{r}{a}\right)-\sin\theta\frac{\mathbf{R}\cdot\dot{\mathbf{R}}}{a^2n}=\frac{r_0}{a}. \qquad \text{(A.3)}$$

The perturbation derivative of the reciprocal of the semimajor axis may be obtained as follows:

$$\frac{1}{a}=\frac{2}{(\mathbf{R}\cdot\mathbf{R})^{\frac{1}{2}}}-\frac{\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}}{\mu}. \qquad \text{(A.4)}$$

Differentiating,

$$\left(\frac{1}{a}\right)_T=-\frac{2\dot{\mathbf{R}}\cdot\mathbf{F}}{\mu}. \qquad \text{(A.4a)}$$

Similarly the perturbation derivative of n_T is given by

$$n_T=-\frac{3}{2}\frac{n}{a}a_T=-\frac{3na}{\mu}\dot{\mathbf{R}}\cdot\mathbf{F}. \qquad \text{(A.4b)}$$

The perturbation derivative of the variable $\mathbf{R}\cdot\dot{\mathbf{R}}/a^2n$ may be obtained as follows:

$$\left(\frac{\mathbf{R}\cdot\dot{\mathbf{R}}}{a^2n}\right)_T=\frac{\mathbf{R}\cdot\mathbf{F}}{a^2n}-\frac{\mathbf{R}\cdot\dot{\mathbf{R}}}{a^2n}\frac{a}{\mu}\dot{\mathbf{R}}\cdot\mathbf{F}. \qquad \text{(A.5)}$$

Substituting the required perturbation derivatives into Eq. (A.2) we may now solve for θ_T.

$$\frac{r_0}{a}\theta_T=-\frac{3na}{\mu}t\dot{\mathbf{R}}\cdot\mathbf{F}+\frac{2r}{\mu}\sin\theta\dot{\mathbf{R}}\cdot\mathbf{F}$$
$$-(\cos\theta-1)\left\{\frac{\mathbf{R}\cdot\mathbf{F}}{a^2n}-\frac{\mathbf{R}\cdot\dot{\mathbf{R}}}{a^2n}\frac{a}{\mu}\dot{\mathbf{R}}\cdot\mathbf{F}\right\}. \qquad \text{(A.6)}$$

Finally,

$$\theta_T=\frac{1-\cos\theta}{anr_0}\mathbf{R}\cdot\mathbf{F}+\frac{g-3t+(r/an)\sin\theta}{anr_0}\dot{\mathbf{R}}\cdot\mathbf{F}. \qquad \text{(A.7)}$$

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The Gravitational Field of the Earth Derived from Motions of Three Satellites

Yoshihide Kozai
Smithsonian Astrophysical Observatory and Harvard College Observatory, Cambridge, Massachusetts

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Coefficients of the second, third, fourth, and fifth harmonics of the earth gravitational field are determined from analysis of motions of three artificial satellites, 1958 $\beta 2$, 1959 η, and 1959 $\iota 1$, as follows:

$$A_2/a_e^2=(1.62329\pm0.00004)\times10^{-3}, \qquad A_4/a_e^4=(9.3\pm0.2)\times10^{-6},$$

$$A_3/a_e^3=(2.29\pm0.02)\times10^{-6}, \qquad A_5/a_e^5=(2.3\pm0.2)\times10^{-7}.$$

I. INTRODUCTION

O'KEEFE, Eckels, and Squires (1959), and Kozai (1959a), have derived coefficients of the zonal harmonics of the earth's gravitational field by analyzing the motion of the first Vanguard artificial satellite, 1958 $\beta 2$. Some observational data of other satellites have subsequently become available to provide a check on the values of these coefficients.

The seventh Explorer satellite, 1959 $\iota 1$, is well suited to this purpose, because its orbital elements are quite different from those of the Vanguard satellite and the air-drag effect is very small.

The motion of 1959 $\iota 1$, during its period of transmission of 108 Mc (October 13 to December 3, 1959), was analyzed (Kozai 1960) by the same method as that used for 1958 $\beta 2$ (Kozai 1959a). However, this interval does not cover the whole period of the periodic inequality (about 106 days) due to the odd-order harmonics. Almost all the observations were made at Minitrack and at Baker-Nunn camera stations.

Table I shows the observed amplitudes of $\sin\omega$ and $\cos\omega$ (ω being the argument of perigee) in the eccentricity and the argument of perigee with values computed with the value $A_3/a_e^3=2.2\times10^{-6}$, as derived from 1958 $\beta 2$.

Differences between the observed and the computed values are appreciable and may show the existence of the fifth-harmonic term of the earth's potential. The amplitudes of the inclination and the longitude of the node are less than their respective standard errors, as expected.

In addition, data for 1959 η, the third Vanguard satellite (Kozai unpublished) have been derived from observations made during the period September 18 to December 10, 1959. However, large systematic deviations in the argument of perigee remain after reduction, so the data for the perigee of this satellite will be excluded from the present analysis.

Table I. Observed and expected amplitudes for 1959 $\iota 1$.

	Eccentricity (e)	Argument of perigee (ω)
Observed amplitude	0.000804± 9	1°183+0°037
Expected amplitude	0.000693±25	1.08 ±0.04

Table II. Mean elements of the three satellites.

	1958 $\beta 2$	1959 η	1959 $\iota 1$
n	10.737127±6	11.062365±5	14.212176±5
e	0.189862±4	0.190035±5	0.036919±7
i	34°2516 ±8	33°3522 ±7	50°3123 ±9
a	1.360510	1.333693	1.128677
T_0	Feb. 25.0, 1959	Oct. 30.0, 1959	Nov. 9.0, 1959

For 1958 $\beta 2$, the orbital elements used in the present paper are not the same in the previous paper (Kozai 1959a), but are derived from observations made during the period January 3 to April 19, 1959.

Table II gives the mean elements of the three satellites; n is the daily anomalistic mean motion in number of revolutions, and a is the semimajor axis expressed in equatorial earth radii. To derive these mean elements all possible perturbations have been subtracted.

To compute the value of the semimajor axis from the anomalistic mean motion, the following formula by Kozai (1959c) is used:

$$n^2a^3=GM\{1-(A_2/p^2)(1-\tfrac{3}{2}\sin^2 i)(1-e^2)^{\frac{1}{2}}\},$$

where the values of $GM=398.618$ (megameters)^3 (kiloseconds)$^{-2}$ (O'Keefe *et al.* 1959) and a_e=6378.388 km are adopted.

II. ODD HARMONICS

If the potential of the earth is expressed by the equation

$$U=\frac{GM}{r}\left\{1+\frac{A_2}{r^2}\left(\frac{1}{3}-\sin^2\delta\right)+\frac{A_3}{r^3}\left(-\frac{3}{2}+\frac{5}{2}\sin^2\delta\right)\sin\delta\right.$$
$$+\frac{A_4}{r^4}\left(\frac{3}{35}+\frac{1}{7}\sin^2\delta-\frac{1}{4}\sin^2 2\delta\right)$$
$$\left.+\frac{A_5}{r^5}\left(\frac{15}{8}-\frac{35}{4}\sin^2\delta+\frac{63}{8}\sin^4\delta\right)\sin\delta\right\},$$

we obtain theoretical expressions for the long-periodic inequalities due to the third and the fifth harmonics

(Brouwer 1959; Kozai 1959a), as follows:

$$\delta e=\frac{3}{4}\frac{\sin i}{A_2 a}\left\{A_3-\frac{5}{4}\frac{A_5}{p^2}\frac{8-28\sin^2 i+21\sin^4 i}{4-5\sin^2 i}(1+\tfrac{3}{4}e^2)\right\}\sin\omega,$$

$$\delta i=-\frac{e}{1-e^2}\cot i\cdot\delta e,$$

$$\delta\omega=\frac{3}{4}\frac{1}{A_2 pe\sin i}\Bigg[A_3(\sin^2 i-e^2\cos^2 i)-\frac{5}{4}\frac{A_5}{p^2}\left\{\frac{8-28\sin^2 i+21\sin^4 i}{4-5\sin^2 i}\sin^2 i\left(1+\frac{e^2}{4}\right)(1+6e^2)-\frac{32-296\sin^2 i+560\sin^4 i-315\sin^6 i}{(4-5\sin^2 i)^2}\times e^2\cos^2 i(1+\tfrac{3}{4}e^2)\right\}\Bigg]\cos\omega,$$

$$\delta\Omega=\frac{3}{4}\frac{e\cot i}{A_2 p}\left\{A_3-\frac{5}{4}\frac{A_5}{p^2}\times\frac{32-296\sin^2 i+560\sin^4 i-315\sin^6 i}{(4-5\sin^2 i)^2}(1+\tfrac{3}{4}e^2)\right\}\cos\omega,$$

where

$$p=a(1-e^2).$$

Although some terms with argument 3ω are derived from the fifth harmonic, their amplitudes, because of small factors, are so small that they may be neglected.

Under the assumption that for the three satellites the amplitudes of the terms with argument ω can be expressed by the above formulae, the following equations of condition hold:

	Observed	Computed I	Computed II
1958 $\beta2$			
e	$0.1910x-0.0731y=0.000444\pm8$	0.000420	0.000444
i	$-3.17x+1.21y=-0\overset{\circ}{.}0082\pm0\overset{\circ}{.}0011$	$-0\overset{\circ}{.}0070$	$-0\overset{\circ}{.}0074$
ω	$55.18x-36.65y=0\overset{\circ}{.}120\pm0\overset{\circ}{.}005$	$0\overset{\circ}{.}118$	$0\overset{\circ}{.}128$
Ω	$5.66x+11.22y=0\overset{\circ}{.}012\pm0\overset{\circ}{.}003$	$0\overset{\circ}{.}015$	$0\overset{\circ}{.}013$
1959 η			
e	$0.1903x-0.0864y=0.000408\pm4$	0.000415	0.000442
i	$-3.27x+1.48y=-0\overset{\circ}{.}0080\pm0\overset{\circ}{.}0010$	$-0\overset{\circ}{.}0071$	$-0\overset{\circ}{.}0076$
Ω	$5.95x+11.18y=0\overset{\circ}{.}012\pm0\overset{\circ}{.}003$	$0\overset{\circ}{.}016$	$0\overset{\circ}{.}014$
1969 $\iota1$			
e	$0.3149x+0.3874y=0.000815\pm8$	0.000810	0.000732
ω	$489.0x+563.9y=1\overset{\circ}{.}18\pm0\overset{\circ}{.}04$	$1\overset{\circ}{.}25$	$1\overset{\circ}{.}14$

where a_e is the earth's equatorial radius,

$$x=\frac{A_3}{a_e^3}\times10^3,\quad y=\frac{A_5}{a_e^5}\times10^3.$$

By assigning weights inversely proportional to the squares of the respective probable errors of the observed values, x and y are obtained by the method of least squares. The results are as follows:

$$A_3/a_e^3=(2.285\pm0.018)\times10^{-6},$$

$$A_5/a_e^5=(2.32\pm0.21)\times10^{-7}.$$

Computed values from these coefficients are shown on the right-hand side of each equation of condition as I.

The consistency is satisfactory. On the other hand, if there is no fifth-harmonic term, the coefficient of the third harmonic becomes 2.324×10^{-6}. Computed amplitudes in this case are written under II. These results indicate that the fifth harmonic terms are significant.

III. EVEN HARMONICS

Coefficients of the second and fourth harmonics can be derived from the observed secular motions of the argument of perigee and the ascending node.

Kozai (1959c) has derived theoretical formulae expressing the secular motions. G. Hori at Yale University Observatory and the present author have checked them with those of Brouwer (1959) and Garfinkel

Table III. Secular motions.

	fA_2^2/p^4	$g\times10^3$	$h\times10^3$	k	Observed	p.e.$\times10^5$	$v\times10^5$
$\dot\omega$ ($\beta2$)	$0\overset{\circ}{.}00173$	$2\overset{\circ}{.}714971$	$0\overset{\circ}{.}1928$	$0\overset{\circ}{.}00056$	$4\overset{\circ}{.}41120$	±11	−5
$\dot\Omega$ ($\beta2$)	0.00014	−1.857624	−0.4349	−0.00040	−3.01986	±7	−11
$\dot\Omega$ (η)	0.00022	−2.012950	−0.5185	−0.00039	−3.27255	±5	3
$\dot\omega$ ($\iota1$)	0.00311	2.092288	−1.2009	0.00018	3.39170	±120	320
$\dot\Omega$ ($\iota1$)	−0.00155	−2.571809	0.0632	−0.00022	−4.17593	+10	3

(1959). The Kozai formulae are:

$$\dot{\omega}=f_1\frac{A_2^2}{p^4}+g_1\frac{A_2}{a_e^2}+h_1\frac{A_4}{a_e^4}+k_1,$$

$$\dot{\Omega}=f_2\frac{A_2^2}{p^4}+g_2\frac{A_2}{a_e^2}+h_2\frac{A_4}{a_e^4}+k_2,$$

where

$$f_1=n(2-\tfrac{5}{2}\sin^2 i)\left\{2+\frac{e^2}{2}-2(1-e^2)^{\frac{1}{2}}\right.$$
$$\left.-\sin^2 i\left(\frac{43}{24}-\frac{e^2}{48}-3(1-e^2)^{\frac{1}{2}}\right)\right\}-\frac{5}{12}e^2 n\cos^4 i,$$

$$g_1=\frac{n}{p^2}(2-\tfrac{5}{2}\sin^2 i)a_e^2,$$

$$h_1=\frac{n}{p^4}\left\{\frac{12}{7}-\frac{93}{14}\sin^2 i+\frac{21}{4}\sin^4 i\right.$$
$$\left.+e^2\left(\frac{27}{14}-\frac{189}{28}\sin^2 i+\frac{81}{16}\sin^4 i\right)\right\}a_e^4.$$

$$f_2=-n\left\{\frac{3}{2}+\frac{e^2}{6}-2(1-e^2)^{\frac{1}{2}}\right.$$
$$\left.-\sin^2 i\left(\frac{5}{3}-\frac{5}{24}e^2-3(1-e^2)^{\frac{1}{2}}\right)\right\}\cos i,$$

$$g_2=-\frac{n}{p^2}a_e^2\cos i,$$

$$h_2=-\frac{n}{p^4}\,\frac{12-21\sin^2 i}{14}\cos i(1+\tfrac{3}{2}e^2)a_e^4.$$

Here all elements appearing in the expressions of f, g, and h are mean ones, which are given in Table II.

Luni-solar effects (Kozai 1959b) are expressed by k_1 and k_2, and for the lunar terms

$$k_1=\frac{3}{4}\frac{n'^2}{n}m'(1-e^2)^{-\frac{1}{2}}(2-\tfrac{5}{2}\sin^2 i+\tfrac{1}{2}e^2)(1-\tfrac{3}{2}\sin^2 i'),$$

$$k_2=-\frac{3}{4}\frac{n'^2}{n}m'(1-e^2)^{-\frac{1}{2}}\cos i(1+\tfrac{3}{2}e^2)(1-\tfrac{3}{2}\sin^2 i'),$$

where

$$\sin^2 i'=\tfrac{1}{2}\sin^2 J(1+\cos^2\epsilon)+\sin^2\epsilon\cos^2 J,$$

and n' is the lunar mean motion, m', the mass of the moon expressed in units of the earth's mass, J, the lunar inclination to the ecliptic, and ϵ, the obliquity. If one sets $m'=1$ and $J=0$, the solar terms are derived.

Values of $f_1A_2^2/p^4$ can be computed with sufficient accuracy by the approximate value of the second harmonic already known. These values with other coefficients f, g, and h are given in Table III.

The observed values are the daily motions in degrees. The coefficients of the second and the fourth harmonics are derived by the method of least squares after assigning to each equation weights that are inversely proportional to the square of the probable error. The results are

$$A_2/a_e^2=(1.623285\pm 0.000036)\times 10^{-3},$$
$$A_4/a_e^4=(9.29\pm 0.18)\times 10^{-6}.$$

Residuals are shown in the last column of Table III as v. Apparently all secular motions can be expressed by the second and the fourth harmonics without the addition of other terms of higher zonal harmonics, within the present limits of accuracy.

The accuracy of these coefficients of even harmonics is usually limited by that of the mean values of the eccentricity and the inclination, but not by that of the secular motions. Also, the relative error of the coefficient of the second harmonic derived here is of the same order of magnitude as those of the earth's equatorial radius and of the value of GM. Therefore, more accurate values of the even harmonics cannot be obtained simply by extending the time interval covered by the observations.

For comparison, the values of the coefficients derived here have been converted into O'Keefe's (1959) notation and units and then compared with the values derived by O'Keefe *et al.* (1959), as shown in Table IV.

TABLE IV. Coefficients of harmonics in megameters–kiloseconds unit.

	$A_{2,0}$	$A_{3,0}$	$A_{4,0}$	$A_{5,0}$
Kozai	−17.5504 ± 4	0.237 ±2	1.40 ±0.03	0.98 ±0.09
O'Keefe *et al.*	−17.555 ±1	0.25 ±0.03	1.12 ±0.04	0.2 ±0.3

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Some Remarks on the Motion of a Satellite of an Oblate Planet

RICHARD B. BARRAR
System Development Corporation, Santa Monica, California

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There has been considerable work recently in trying to compute analytically with great accuracy the effects of the gravitational field of an oblate planet on the motion of a satellite. Recently, Vinti (1959) has given a very accurate answer by treating the problem in oblate spheroidal coordinates. The present paper derives by an entirely different method a result that has essentially the same accuracy as that of Vinti's but is in a simpler form. The form of the answer resembles very closely that given by Garfinkel (1958).

I. INTRODUCTION

STERNE (1957, 1958) pointed out that in spherical coordinates, any Hamiltonian having a potential of the form

$$V=f_1(r)+[g_2(\theta)/r^2] \qquad (1)$$

[where $f_1(r)$ is any function of the radius vector r, and $g_2(\theta)$ is any function of θ, the complement of the polar angle] can be reduced to quadratures, while the potential

$$V_{\text{exact}}=-\frac{1}{r}+\frac{2kP_2(\sin\theta)}{r^3} \qquad (2)$$

"contains the potential terms that are overwhelmingly important in comparison with all others arising from gravitational forces." In Eq. (2) we have used the units in dimensionless form as given in Garfinkel (1958).

Garfinkel (1959) reviews the work he and Sterne have done in fitting a potential in the form (1) to that of the form (2).

Recently, Vinti (1959) has shown that if one works in oblate spheroidal coordinates, he can reduce the Hamiltonian whose potential is

$$V_{\text{Vinti}}=-\frac{1}{r}\sum_{n=0}^{\infty}(-1)^n(2k)^n\frac{P_{2n}(\sin\theta)}{r^{2n}} \qquad (3)$$

to quadratures.

Since the first two terms of V_{Vinti} are the same as those of V_{exact} and since $k\sim 0.55\times 10^{-3}$, Vinti's solution is undoubtedly an excellent one. However, Vinti only reduced the solution to quadratures. His result is four elliptic integrals of the third kind.

In this paper we introduce a potential V^* such that

$$V^*=-\frac{1}{r}+\frac{2kP_2(\sin\theta)}{r^3}+k^{\frac{3}{2}}0\left(\frac{1}{r^4}\right). \qquad (4)$$

That is, our potential V^* agrees with the first two terms of V_{exact}; moreover, it introduces an error of about $(k^{\frac{1}{2}}/r)$ times the second term in V_{exact}.

With our potential we can now integrate the Hamiltonian exactly in spherical coordinates. Also, our solution has a property similar to Garfinkel (1958) in that it only involves two elliptic integrals.

By utilizing the remarks at the end of the paper we can considerably reduce the error term, namely, to $0(k^2)$ or about the same accuracy as that of Vinti.

II. THE POTENTIAL V*

If one is in Cartesian coordinates (x, y, z) and corresponding spherical coordinates (r, θ, ϕ), and now moves his center of coordinates to $(0, 0, c)$ (see Fig. 1), he introduces new Cartesian coordinates (x, y, z') with $z'+c=z$, and new spherical coordinates (r_2, θ', ϕ) with

$$z'=r_2\sin\theta', \quad r_2=[x^2+y^2+(z')^2]^{\frac{1}{2}}.$$

We prove below the fundamental identity

$$-\frac{1}{r_2}+\frac{(2k)^{\frac{1}{2}}P_1(\sin\theta')}{r_2{}^2}=-\frac{1}{r}+\frac{2kP_2(\sin\theta)}{r^3}$$
$$+\sum_{n=3}^{\infty}(n-1)(2k)^{n/2}\frac{P_n(\sin\theta)}{r^{n+1}}, \qquad (5)$$

where

$$c=(2k)^{\frac{1}{2}}.$$

Before proving (5) we note that it fulfills all the properties of our potential V^*; moreover, in the spherical coordinate system (r_2, θ', ϕ) it is of the class (1), hence, is reducible to quadratures.

Proof of (5). We borrow from Stratton (1941, pp. 173 174) the formulae

$$\frac{1}{r_2}=\sum_{n=0}^{\infty}c^n\frac{P_n(\sin\theta)}{r^{n+1}}, \qquad (6)$$

$$\frac{P_n(\sin\theta)}{r^{n+1}}=\frac{(-1)^n}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right). \qquad (7)$$

We first substitute (7) in (6) to obtain

$$\frac{1}{r^2}=\sum_{n=0}^{\infty}c^n\frac{(-1)^n}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right). \qquad (8)$$

Now, operating on the left-hand side of (8) with $-\partial/\partial z'$ and on the right-hand side with $-\partial/\partial z$ (since the two are equivalent) and applying formula (7) we

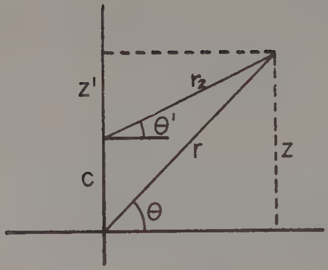

FIG. 1. r_2, θ' coordinates.

obtain

$$\frac{P_1(\sin\theta')}{r_2^2} = -\frac{\partial}{\partial z'}\left(\frac{1}{r_2}\right) = \sum_{n=0}^{\infty} c^n \frac{(-1)^{n+1}}{n!}\frac{\partial^{n+1}}{\partial z^{n+1}}\left(\frac{1}{r}\right)$$

$$= \sum_{n=0}^{\infty} c^n \frac{(n+1)!}{n!}\frac{P_{n+1}(\sin\theta)}{r^{n+2}}$$

$$= \sum_{n=0}^{\infty} c^n (n+1)\frac{P_{n+1}(\sin\theta)}{r^{n+2}}. \qquad (9)$$

Finally setting $c=(2k)^{\frac{1}{2}}$ and combining (6) and (9) we obtain (5). [It should be remarked although we obtained the expansion (5) purely formally, that since both sides of Eq. (5) are solutions of Laplace's equation, the rearrangement and differentiation of the series can be made perfectly rigorous.]

III. SOLUTION OF HAMILTON-JACOBI EQUATIONS WITH POTENTIAL V*

In (r_2, θ', ϕ) coordinates and with the potential V^* we obtain the following Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial S}{\partial r_2}\right)^2+\frac{1}{r_2^2}\left(\frac{\partial S}{\partial \theta'}\right)^2+\frac{1}{r_2^2\cos^2\theta'}\left(\frac{\partial S}{\partial \phi}\right)^2\right] -\frac{1}{r_2}+\frac{c\sin\theta'}{r_2^2}=0. \qquad (10)$$

Now following, for example, the derivation of Garfinkel (1958), we first find that S can be written as

$$S=-\alpha_1 t+\alpha_3\phi+\int p_1 dr_2+\int p_2 d\theta', \qquad (11)$$

where

$$p_1=\left[2\alpha_1+\frac{2}{r_2}-\frac{\alpha_2^2}{r_2^2}\right]^{\frac{1}{2}},$$

$$p_2=[\alpha_2^2-\alpha_3^2\sec^2\theta'-2c\sin\theta']^{\frac{1}{2}}.$$

The solutions of the equations of motion $\partial S/\partial\alpha_i=\beta_i$; $i=1, 2, 3$ become

$$\int\frac{dr_2}{p_1}=\beta_1+t,$$

$$-\int\frac{\alpha_2 dr_2}{r_2^2 p_1}+\int\frac{\alpha_2 d\theta'}{p_2}=\beta_2, \qquad (12)$$

$$\phi-\int\frac{\alpha_3\sec^2\theta'\, d\theta'}{p_2}=\beta_3.$$

Now set

$$\alpha_1=-\tfrac{1}{2}a, \quad \alpha_2=p=a(1-e^2), \quad i=\min\theta',$$

$$n^2a^3=1, \quad \alpha_3=\alpha_2\cos i, \quad \beta_1=\sigma/n. \qquad (13)$$

Then in (12) the transformations $r_2=a(1-e\cos E)$ and $r_2=p/(1+e\cos v)$, where E and v are new variables, reduce the two r_2 integrals, respectively, to

$$\int_{\min r}^{r}\frac{dr_2}{p_1}=\frac{E-e\sin E}{n}, \quad \int_{\min r}^{r}\frac{\alpha_2 dr_2}{r_2^2 p_1}=v. \qquad (14)$$

The θ' integrals are a little harder to handle than those treated by Garfinkel, but in the end they reduce to essentially the same thing.

First, the substitution $x=\sin\theta'$ reduces the first θ' integral to

$$\int\frac{\alpha_2 d\theta'}{p_2}=\int\frac{dx}{[F(x)]^{\frac{1}{2}}}$$

$$=\int\frac{dx}{\left[\left(\frac{\alpha_1^2-\alpha_3^2}{\alpha_2^2}-x^2\right)-\frac{2cx(1-x^2)}{\alpha_2^2}\right]^{\frac{1}{2}}}. \qquad (15)$$

The third-degree polynomial $F(x)$ for $c>0$ and

$$(\alpha_2^2-\alpha_3^2)/\alpha_2^2=\sin^2 i$$

behaves as shown in Fig. 2. This follows from its following values:

$$\begin{aligned} F(-\infty)&=-\infty,\\ F(-1)&=\sin^2 i-1<0,\\ F(0)&=\sin^2 i>0,\\ F(1)&=\sin^2 i-1<0,\\ F(\infty)&=\infty. \end{aligned} \qquad (16)$$

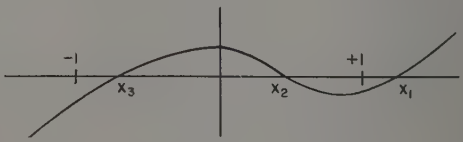

FIG. 2. Plot of $F(x)$.

Thus, its roots x_1, x_2, x_3 can be characterized by

$$x_1>1>x_2>0>x_3>-1, \qquad F(x)=\frac{2c}{\alpha_2^2}(x-x_1)(x-x_2)(x-x_3). \tag{17}$$

The region of physical interest is between the roots x_3 and x_2. Thus, using the formulas in Gröbner-Hofreiter (1957, p. 78), one sets

$$\begin{aligned} k^2&=(x_2-x_3)/(x_1-x_2)\\ \sin\theta'=x&=(x_2-x_3)\sin^2\bar{\phi}+x_3\\ &=(x_2-x_3)\,\mathrm{sn}^2(u,k)+x_3. \end{aligned} \tag{18}$$

The first θ' integral becomes

$$\int\frac{\alpha_2 d\theta'}{p_2}=\int\frac{dx}{[F(x)]^{\frac{1}{2}}}=\frac{2\alpha_2}{[2c(x_1-x_3)]^{\frac{1}{2}}}\int_0^{\bar{\phi}}\frac{d\bar{\phi}}{[1-k^2\sin^2\bar{\phi}]^{\frac{1}{2}}}=\frac{2\alpha_2 u}{[2c(x_1-x_3)]^{\frac{1}{2}}}. \tag{19}$$

The second θ' integral becomes

$$\begin{aligned}\int\frac{\alpha_3\sec^2\theta'\,d\theta'}{p_2}&=\frac{\alpha_3}{\alpha_2}\int\frac{dx}{(1-x^2)[F(x)]^{\frac{1}{2}}}\\ &=\frac{2\alpha_3}{[2c(x_1-x_3)]^{\frac{1}{2}}}\left[\frac{1}{(1-x_3)}\int_0^{\bar{\phi}}\frac{d\bar{\phi}}{\left\{1-\left(\frac{x_2-x_3}{1-x_3}\right)\sin^2\bar{\phi}\right\}(1-k^2\sin^2\bar{\phi})^{\frac{1}{2}}}\right.\\ &\qquad\left.+\frac{1}{(1+x_3)}\int_0^{\bar{\phi}}\frac{d\bar{\phi}}{\left\{1+\left(\frac{x_2-x_3}{1+x_3}\right)\sin^2\bar{\phi}\right\}(1-k^2\sin^2\bar{\phi})^{\frac{1}{2}}}\right]\\ &=\frac{2\alpha_3}{[2c(x_1-x_3)]^{\frac{1}{2}}}\left[\frac{1}{(1-x_3)}\int_0^{u}\frac{du}{\left\{1-\left(\frac{x_2-x_3}{1-x_3}\right)\mathrm{sn}^2(u,k)\right\}}+\frac{1}{(1+x_3)}\int_0^{u}\frac{du}{\left\{1+\left(\frac{x_2-x_3}{1+x_3}\right)\mathrm{sn}^2(u,k)\right\}}\right].\end{aligned} \tag{20}$$

Two remarks should be made about the integrals (19) and (20). First, if one solves the cubic equation $F(x)$ for x_1, he finds $x_1\sim\alpha_2^2/2c$. Thus, the factor $1/[2c(x_1-x_3)]^{\frac{1}{2}}\sim p$. Secondly, from (18) it is seen that when $u=\bar{\phi}=0$, θ' is x_3, and also $x_3\sim-\sin i$, or when u goes from 0 to k, θ' goes from approximately $-\sin i$ to approximately $\sin i$, not from 0 to $\sin i$ as one would have in Garfinkel's paper.

Garfinkel (1958) gives two ways of accurately evaluating the elliptic integrals (19) and (20). I will only add the reference to the recent work of Ward (1960), where he evaluates complete elliptic integrals of the third kind as found in (20).

The complete set of formulae is

$$\begin{aligned} E-e\sin E&=nt+\sigma,\\ r_2=a(1-e\cos E)&=p/(1+e\cos v),\\ \tan\frac{v}{2}&=\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\tan\frac{E}{2},\\ (v+\omega')&=\frac{2pu}{[2c(x_1-x_3)]^{\frac{1}{2}}} \end{aligned} \tag{21}$$

(when $v=-\omega'$, $u=0$, hence ω' is an angle in the plane of the orbit from perigee to the point having the lowest θ')

$$\sin\theta'=(x_2-x_3)\,\mathrm{sn}^2(u,k)+x_3,$$

and

$$\phi-\Omega'=\frac{2\alpha_3}{[2c(x_1-x_3)]^{\frac{1}{2}}}\times\left[\frac{1}{(1-x_3)}\int_0^u\frac{du}{1-\left(\frac{x_2-x_3}{1-x_3}\right)\mathrm{sn}^2(u,k)}+\frac{1}{1+x_3}\int_0^u\frac{du}{1+\left(\frac{x_2-x_3}{1+x_3}\right)\mathrm{sn}^2(u,k)}\right],$$

where Ω' is the longitude when θ' is at its lowest value

$$\begin{aligned} r&=[r_2^2+c^2+2r_2c\sin\theta']^{\frac{1}{2}},\\ z&=c+r_2\sin\theta',\\ \sin\theta&=z/r,\\ c&=(2k)^{\frac{1}{2}}. \end{aligned}$$

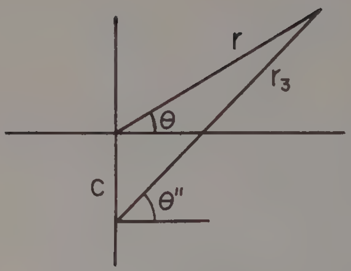

FIG. 3. r_3, θ'' coordinates.

IV. MAGNITUDE OF THE ERROR

Instead of using the coordinates r_2, θ' introduced in Fig. 1, we can also use the coordinates r_3, θ'' shown in Fig. 3.

It is easily seen that an inversion $z \rightarrow -z$ or equivalently $\theta \rightarrow -\theta$ takes the configuration of Fig. 2 into that of Fig. 3 and vice versa; hence, we can apply the results obtained in the (r_2, θ') coordinates to those of the (r_3, θ'') coordinates. In particular, one has the expansion

$$-\frac{1}{r_3}+\frac{(2k)^{\frac{1}{2}}}{r_3^{\,2}}P_1(-\sin\theta'')=-\frac{1}{r}+\frac{2kP_2(-\sin\theta)}{r^3}+\sum_{n=3}^{\infty}(n-1)(2k)^{n/2}\frac{P_n(-\sin\theta)}{r^{n+1}}. \qquad (5')$$

It should be remarked, however, that in the above inversion we have changed $\theta_{\min}'$ to $\theta_{\max}''$ so that the formulae (21) will have the following changes:

$$\sin\theta''=(x_3-x_2)\,\mathrm{sn}^2(u,k)-x_3; \quad x_2>x_3,$$

$$(v+\omega'')=\frac{2pu}{[2c(x_1-x_3)]^{\frac{1}{2}}}.$$

$$\phi-\Omega''=\frac{2\alpha_3}{[2c(x_1-x_3)]^{\frac{1}{2}}}\times\left[\frac{1}{(1-x_3)}\int_0^u\frac{du}{1-\left(\dfrac{x_2-x_3}{1-x_3}\right)\mathrm{sn}^2(u,k)}+\frac{1}{1+x_3}\int_0^u\frac{du}{1+\left(\dfrac{x_2-x_3}{1+x_3}\right)\mathrm{sn}^2(u,k)}\right], \qquad (21')$$

where ω'' and Ω'' are measured from $\theta_{\max}''$; (x_1, x_2, x_3) are still the roots of equation $F(x)$ of (15).

One has the identity

$$P_n(-\sin\theta)=(-1)^nP_n(\sin\theta).$$

Thus, since the main error term in the potential of (5) that is in (r_2, θ') coordinates is $[+2(2k)^{\frac{3}{2}}/r^4]P_3(\sin\theta)$, in (r_3, θ'') coordinates it is $[-2(2k)^{\frac{3}{2}}/r^3]P_3(\sin\theta)$. Hence, by comparing the answers obtained by using the potential (5) with those obtained by using the potential (5′), one has a good way of estimating the error. Moreover, by taking an average of the two answers one should be able to considerably reduce any error.

In order to prove in a more rigorous fashion that the average of the two solutions (21) and (21′) will in fact remove the term of $0(k^{\frac{3}{2}})$ one can proceed as follows: Write the potential as

$$-\frac{1}{r}+2k\frac{P_2(\sin\theta)}{r^3}+2\eta\frac{P_3(\sin\theta)}{r^4}+\sum_{n=4}^{\infty}(n-1)\eta^m(2k)^q\frac{P_n(\sin\theta)}{r^{n+1}}, \qquad (22)$$

where in the summation term $n=3m+q$ for some suitably chosen positive integers m and q.

Then the two solutions (21) and (21′) obtained in this paper correspond to solutions of the system of equations

$$d\mathbf{x}_i/dt=f_i(\mathbf{x},\eta,k) \quad i=1,\cdots,6$$

$$\mathbf{x}(t_0)=\mathbf{x}_0$$

for

$$\eta=\pm(2k)^{\frac{3}{2}},$$

where $\mathbf{x}$ is a six-component vector comprising the three Cartesian components of position and the three Cartesian components of velocity and f_i are the terms obtained in the equations of motion using the potential energy (22). [The notation is that of Lefschetz (1948).]

Since the f_i are analytic functions of η, k and $\mathbf{x}$, it follows from the Poincaré expansion theorem [see Lefschetz (1948, p. 35)] that the $x_i(t)$ can be written as

$$x_i(t)=g_i(t,\eta,k,\mathbf{x}_0) \quad i=1,\cdots,6,$$

where the functions g_i are analytic in all their arguments. Hence, we can write

$$\mathbf{x}(t)=\mathbf{g}_0(t,k,x_0)+\eta\mathbf{g}_1(t,k,x_0)+0(\eta^2).$$

Now since the two solutions (21) and (21′) correspond to $\eta=\pm(2k)^{\frac{3}{2}}$ the average of the two solutions will not contain the term $\eta\mathbf{g}_1(t,k,x_0)$; hence, this average solution would correspond to a potential of the form

$$-\frac{1}{r}+2k\frac{P_2(\sin\theta)}{r^3}+0(\eta^2)+0(k^2), \qquad (23)$$

which is the assertion of the paper. Unfortunately, I have not been able to carry this proof over to show that one can also take the average of the constants of the motion (i.e., a, e, ω', etc.) to obtain the same results.

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The Reduction from Geocentric to Geodetic Coordinates

JOHN MORRISON AND SAMUEL PINES
Republic Aviation Corporation, Farmingdale, L. I., New York

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This paper outlines an explicit procedure for obtaining the geodetic (latitude and altitude) coordinates of a point given the geocentric equatorial polar coordinates. The equations relating these coordinate systems require the solution of an eighth-order polynomial. The method outlined herein avoids an iterative procedure for solving these equations by providing a closed-form explicit solution using the Lagrange expansion formula.

INTRODUCTION

It is often required to determine the geodetic coordinates of a point in space in terms of the altitude above the earth's surface and the geodetic latitude from coordinates given in terms of the geocentric radius and geocentric latitude. The transformation relating these coordinates involves the solution of an eighth-order trignometric polynomial. Various procedures for obtaining a solution of these equations by iteration techniques exist. *The American Ephemeris and Nautical Almanac* (1959, p. xvi; 1960, p. 496) contains a solution of the inverse problem in explicit form for a point on the earth's surface. The method used in this report is based on the Lagrange expansion formula. The solution for the geodetic latitude is given as a Fourier series of even multiples of the geocentric latitude with coefficients which are functions of the earth's eccentricity and the polar geocentric distance.

NOTATION

ρ	Geocentric polar distance measured in earth radii.
ϕ'	Geocentric latitude.
h	Geodetic altitude above the earth's surface measured in earth radii.
ϕ	Geodetic latitude.
a, e	Semimajor axis and eccentricity of the earth's meridional ellipse.

THE SOLUTION OF GEODETIC LATITUDE PROBLEM

The earth's surface may be represented by revolving an ellipse of eccentricity, e, and major axis, $a=1$, about the minor polar axis. The geocentric coordinates (ρ, ϕ') of a point are given in terms of the distance measured from the center of the earth and the latitude angle measured from the equatorial plane. The geodetic coordinates (h, ϕ) of the same point are measured in terms of the shortest altitude from the point to the earth's surface with the latitude given by the angle between this normal to the earth's surface and the equatorial plane. The equations relating these coordinates are given by

$$\rho\cos\phi' = (h+c)\cos\phi, \quad \rho\sin\phi' = (h+s)\sin\phi, \tag{1}$$

where

$$c = \{1-e^2\sin^2\phi\}^{-\frac{1}{2}}, \quad s = (1-e^2)c. \tag{1a}$$

Equation (1) may be solved for the tangent of the geodetic latitude

$$\tan\phi = \tan\phi' + \frac{e^2}{\rho\cos\phi'}\,\frac{\sin\phi}{(1-e^2\sin^2\phi)^{\frac{1}{2}}}. \tag{2}$$

A solution of Eq. (2) may be obtained by the Lagrange expansion formula (Whittaker and Watson, pp. 132–133). Let y be given as a function of x as follows:

$$y = x + b(x)g(y), \tag{3}$$

where $b(x)$ is small. Then a function, $f(y)$ which has a Taylor series representation may be written as follows:

$$f(y) = f(x) + \sum_{k=1}^{\infty} \frac{b^k(x)}{k!}\,\frac{d^{k-1}}{dx^{k-1}}\left[\frac{d\,f(x)}{dx}g^k(x)\right]. \tag{4}$$

Let

$$\begin{gathered} y = \tan\phi, \quad f(y) = \tan^{-1} y = \phi, \\ g(y) = \frac{y}{[1+(1-e^2)y^2]^{\frac{1}{2}}} = \frac{\sin\phi}{(1-e^2\sin^2\phi)^{\frac{1}{2}}}, \end{gathered} \tag{5}$$

and

$$x=\tan\phi',$$
$$b(x)=e^2(1+x^2)^{\frac{1}{2}}/\rho=e^2/\rho\cos\phi', \qquad (5a)$$
$$df/dx=1/(1+x^2).$$

On substituting Eqs. (5) and (5a) into Eq. (4) the solution of the geodetic latitude is given by

$$\phi=\phi'+a_2(e,\rho)\sin2\phi'+a_4(e,\rho)\sin4\phi'+a_6(e,\rho)\sin6\phi'+a_8(e,\rho)\sin8\phi'. \qquad (6)$$

The coefficients in Eq. (6) are given by

$$a_2=\frac{1}{1024\rho}\{512e^2+128e^4+60e^6+35e^8\}+\frac{1}{32\rho^2}\{e^6+e^8\}-\frac{3}{256\rho^3}\{4e^6+3e^8\},$$

$$a_4=\frac{-1}{1024\rho}\{64e^4+48e^6+35e^8\}+\frac{1}{16\rho^2}\{4e^4+2e^6+e^8\}+\frac{15e^8}{256\rho^3}-\frac{e^8}{16\rho^4}, \qquad (7)$$

$$a_6=\frac{3}{1024\rho}\{4e^6+5e^8\}-\frac{3}{32\rho^2}\{e^6+e^8\}+\frac{35}{768\rho^3}\{4e^6+3e^8\},$$

$$a_8=\frac{e^8}{2048}\left\{-\frac{5}{\rho}+\frac{64}{\rho^2}-\frac{252}{\rho^3}+\frac{320}{\rho^4}\right\}.$$

A similar expression for the geodetic latitude in terms of the so-called oblateness factor ϵ may be obtained by replacing the earth's eccentricity e by

$$e^2=2\epsilon-\epsilon^2. \qquad (8)$$

The expressions in Eq. (7) are sufficient for eight-digit accuracy since the value of e^2 is given by 0.00672267002, if $\epsilon=1/297$.

THE GEODETIC ALTITUDE

Several expressions for the altitude h may be obtained once the geodetic latitude is known.

$$h=\rho\frac{\cos'\phi}{\cos\phi}-\frac{1}{(1-e^3\sin^2\phi)^{\frac{1}{2}}},$$
$$h=\rho\frac{\sin\phi'}{\sin\phi}-\frac{1-e^2}{(1-e^2\sin^2\phi)^{\frac{1}{2}}}, \qquad (9)$$
$$h=\rho\cos(\phi-\phi')-(1-e^2\sin^2\phi)^{\frac{1}{2}}.$$

All the above expressions have different regions of applicability for obtaining the best accuracy for given values of the latitude. All three, however, suffer from a common defect for altitudes less than one earth radius. For ρ close to one, h is given by the difference of two comparable numbers and losses in digit accuracy will unavoidably occur. For all other ranges, Eq. (9) suffices to determine the geodetic altitude.

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Astrometric Investigations of Galactic Clusters. I. NGC 129

Alan P. Lenham*
Yerkes Observatory, Williams Bay, Wisconsin

AND

Otto G. Franz
U. S. Naval Observatory, Washington, D. C.

(Received October 10, 1960)

From two pairs of plates taken with the 40-inch refractor of the Yerkes Observatory with an epoch difference of 42 years, the relative proper motions of 70 stars in the region of NGC 129 have been determined. The total number of cluster members brighter than $V=12.4$ was found to be about 40 stars within 20 minutes of arc from the apparent cluster center. On the basis of its proper motion it is shown that the Cepheid DL Cas has a very high probability of being a cluster member.

I. INTRODUCTION

THE first two decades of the Yerkes 40-inch refractor program included plates of a number of galactic clusters. Second epoch plates of several clusters were taken after a lapse of some 20 years and the motions of cluster stars and field stars were derived (Ebbighausen).

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* Now at Royal Military College of Science, Shrivenham, Near Swindon, Wilts, England.

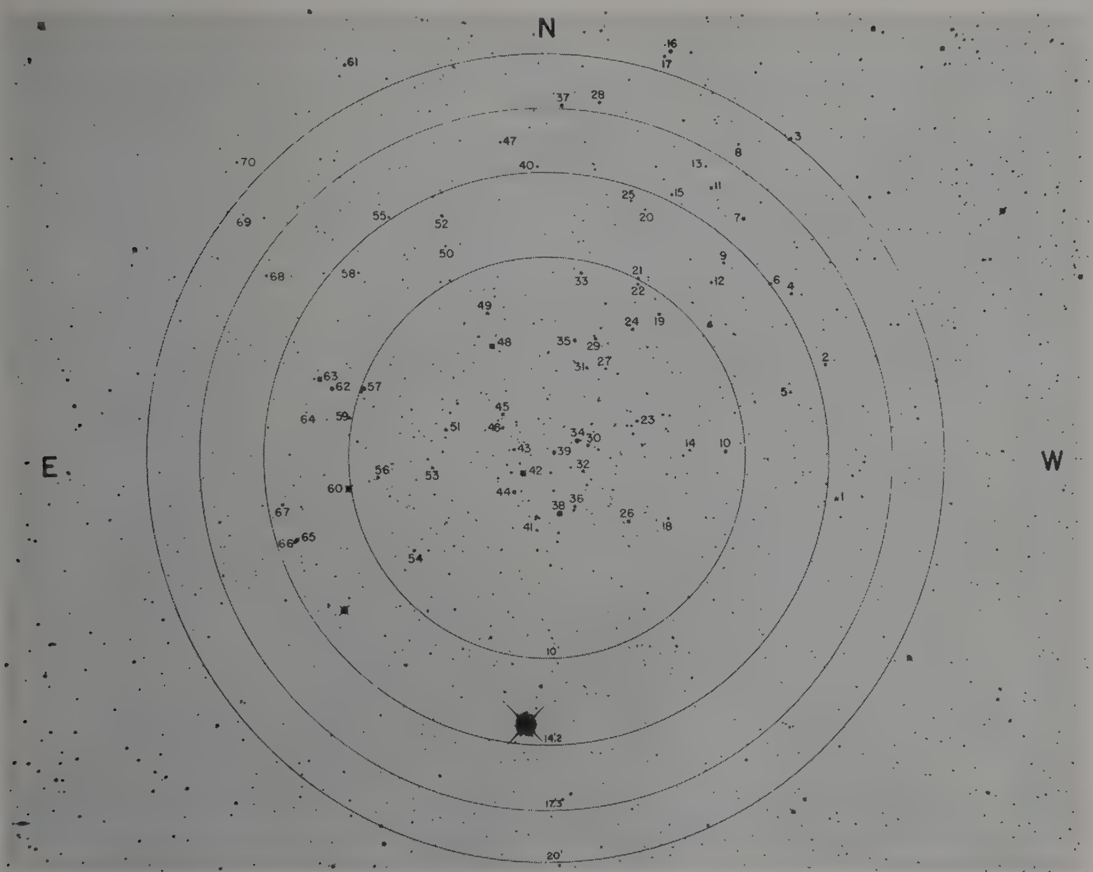

FIG. 1. Stars measured in the region of NGC 129. Photograph taken by A. A. Hoag with the 40-inch reflector, U. S. Naval Observatory, Flagstaff, Arizona.

However, a number of interesting clusters remained for which second epoch plates had not been obtained. In 1958 a card catalogue was compiled of the early cluster plates and second epoch plates were taken as part of the regular astrometric program. The time interval between the first and the second plates on this new program was, on the average, of the order of 40 years. Every effort was made to choose a plate–filter combination closely approximating that of the first epoch plates (Strand 1958) in order to avoid systematic color errors which might affect the derived proper motions.

This paper presents the relative proper motions of 70 stars in the field of the cluster NGC 129, R.A. $00^{\mathrm{h}}24^{\mathrm{m}}3$, Dec. $+59°40'$ (1900). An identification chart for these stars is reproduced in Fig. 1.

II. PLATE MATERIAL AND REDUCTIONS

The observational data for the plates used in this investigation are given in Table I.

The plates were measured (by A.P.L.) in x and y, both direct and reversed, on the Gaertner long-screw measuring machine at the Yerkes Observatory. Since plate π 21206, which contains a trail, was used as standard plate, all measured coordinates were oriented to the equator of 1958.669. The reductions of the measures were made on the IBM 650 computer of the U. S. Naval Observatory. A program was written (by O.G.F.) which, in its present form, can be used to derive proper motions for up to 1000 stars, measured on 100 plates with a maximum epoch difference of 100 years.

The reductions are carried out in several separate steps. First the measured coordinates of the standard plate are corrected for differential refraction and

TABLE I. Observational data.

Plate No.	Epoch	Hour angle in minutes (at start)	Exposure time (minutes)	Temperature (°C)
π 2763	1916.714	7 *E*	28	+9.8
π 2892	1916.837	21 *E*	43	+12.2
π 21206	1958.669	54 *E*	8[a]	+15
π 21222	1958.699	39 *E*	8	+16

[a] Two exposures of eight minutes each.

reduced to the equinox of 1950.0 by applying differential aberration and precession.

From one pair of plates with large epoch difference preliminary proper motions are then obtained to allow exclusion of all obvious foreground stars from the final system of reference stars. Of the remaining stars all those measured on all plates are used for the final frame of reference. The number of reference stars in the case of NGC 129 was 64 out of a total of 70 stars measured in the region of the cluster.

By means of least-square solutions, carried out separately in x and y, plate constants, including the quadratic terms, together with their respective weights, are obtained for each plate. With these plate constants the measured coordinates of all stars are then transformed to the corrected coordinate system of the standard plate. Thus we obtain for every star a set of coordinates for each plate epoch which is free of refraction effects and reduced to the equator of 1950.0.

From these sets of reduced coordinates relative proper motions μ_x and μ_y and rectangular coordinates for the equinox and epoch of 1950.0 as well as their respective mean errors are derived by separate least-squares solutions in x and y for each star. All reductions are carried out in floating decimal point routine. The total computing time for 100 stars, measured on four plates, is of the order of 35 minutes.

III. RELATIVE PROPER MOTIONS

The relative proper motions of 70 stars in the region of NGC 129 are given in Table II. The first column contains consecutive identification numbers. Column 2 gives the identification numbers and designations by Arp, Sandage, and Stephens (1959). Columns 3 and 4 give in millimeters the rectangular coordinates x and y, respectively, for the equinox and epoch of 1950.0, with an arbitrary zero point. The scale value is

$$1\text{ mm} = 10''.655.$$

Columns 5, 6, 7, and 8 give the relative proper motions in ten-thousands of a second of arc per year for the

TABLE II. Relative proper motions.

No. (1)	A (2)	x (3)	y (4)	μ_x (5)	$\epsilon\mu_x$ (6)	μ_y (7)	$\epsilon\mu_y$ (8)	Membership (9)
1		−82.612	−42.330	− 22	±19	+ 18	±10	highly probable
2		−79.925	− 4.330	+ 44	6	− 42	8	nonmember
3		−70.739	+60.119	− 54	16	+ 29	9	
4		−70.432	+15.895	+ 27	3	− 27	11	
5		−69.787	−12.052	− 6	19	+ 17	3	highly probable
6		−64.263	+18.730	− 77	4	+ 5	18	nonmember
7		−56.757	+37.221	+ 2	11	− 7	4	highly probable
8		−55.542	+58.358	+ 11	13	− 14	3	
9		−50.919	+24.724	+ 26	4	+ 21	3	
10		−50.856	−28.771	− 31	10	0	10	
11		−47.562	+45.951	+ 77	11	− 6	7	nonmember
12		−47.326	+19.083	0	4	+ 2	9	highly probable
13		−46.048	+52.139	− 74	5	− 48	6	nonmember
14		−40.636	−28.398	+ 9	6	− 65	6	nonmember
15		−36.166	+44.123	+ 25	23	+ 6	10	
16		−36.119	+85.018	− 2	...	− 5	...	highly probable
17		−34.402	+85.528	− 214	...	+ 15	...	nonmember
18		−34.347	−47.638	+ 598	20	+338	8	nonmember
19		−32.178	+10.158	− 52	5	+ 25	10	
20		−28.382	+39.901	− 2	3	+ 6	6	highly probable
21		−26.437	+20.421	+ 580	25	− 64	6	nonmember
22		−26.168	+18.720	+ 90	8	+ 61	15	nonmember
23	151	−25.586	−20.104	− 18	8	0	3	highly probable
24		−24.549	+ 5.879	+ 4	4	− 6	11	highly probable
25		−24.416	+42.411	+ 32	16	+ 27	7	
26	D	−22.939	−48.357	+ 12	11	− 15	3	
27	135	−20.197	− 6.658	+ 46	12	+108	7	nonmember
28		−15.498	+70.198	+ 7	8	− 28	6	
29	123	−13.874	+ 3.283	− 4	10	+ 11	19	highly probable
30	125	−11.417	−26.842	+ 31	20	+ 26	5	
31	118	−11.321	− 4.938	+ 4	12	− 1	19	highly probable
32	121	−10.010	−34.124	− 15	4	+ 37	5	
33		− 9.969	+21.938	+ 7	8	− 9	10	highly probable
34	DL Cas	− 8.416	−25.540	− 5	9	− 16	9	highly probable
35	111	− 7.890	+ 2.821	− 51	7	− 17	12	

equinox 1950.0 with their respective mean errors. The remarks concerning cluster membership, contained in the last column, are explained in detail in Sec. V. All stars, with the exception of Nos. 16, 17, and 61, have been measured on all four plates. An evaluation of the accuracy of the yearly proper motions shows that on the average the mean error for a yearly proper motion determined from two pairs of plates is $\pm 0''.0013/a$ and $\pm 0''.0011/a$ in μ_x and μ_y, respectively. From these values we obtain mean errors of $\pm 0''.0018/a$ and $\pm 0''.0016/a$ for proper motions in x and y determined from one pair of plates with an epoch difference of about 40 years, in close agreement with the accuracy obtained in previous investigations using Yerkes 40-inch plates (Strand 1958).

IV. MEAN RELATIVE CLUSTER MOTION

Three different groups of stars, each within a radius of about $4'.5$, were selected, and their mean relative motions as well as the corresponding standard deviations were obtained. Group A consists of 12 stars within the positional limits $-25<x<+25$ and $-55<y<-5$ (x and y in mm) around the apparent cluster center. The identification numbers of the stars in this group are 30, 32, 34, 36, 38, 39, 41, 43, 44, 45, 46, and 51. Excluded was star No. 42 which, from its large proper motion, definitely appears to be a foreground star. Group B, centered around star No. 13 at an approximate distance of $15'$ from the apparent cluster center, consists of eight stars with numbers 3, 7, 8, 11, 13, 15, 20, and 25. Group C contains 10 stars, namely, numbers 56, 57, 59, 60, 62, 63, 64, 65, 66, and 67, located east of the apparent cluster center at a mean distance of approximately $12'$. The mean relative motions and the standard deviations for these three groups of stars are in seconds of arc per year:

	Mean relative motion		Standard deviations	
Group A	−0.0007	+0.0002	±0.0021	±0.0022
Group B	+0.0002	−0.0001	±0.0045	±0.0023
Group C	+0.0001	−0.0008	±0.0032	±0.0036

An inspection of the mean motions derived for these different groups of stars clearly shows that their values

TABLE II. (Continued)

No. (1)	A (2)	x (3)	y (4)	μ_x (5)	$\epsilon\mu_x$ (6)	μ_y (7)	$\epsilon\mu_y$ (8)	Membership (9)
36	115	− 7.627	−44.097	+ 38±	14	− 25±	4	
37		− 4.667	+69.346	+ 32	16	0	3	
38	A	− 3.210	−46.095	− 5	11	− 46	19	
39	105	− 1.764	−28.866	− 9	3	+ 11	3	highly probable
40		+ 2.399	+52.073	+ 25	12	+ 31	12	
41	96	+ 3.284	−50.641	− 13	6	+ 9	8	highly probable
42	B	+ 7.044	−34.703	+1762	8	+886	3	nonmember
43	71	+ 9.622	−27.969	− 44	13	+ 14	2	
44	C	+ 9.704	−39.968	− 12	3	+ 6	19	highly probable
45	61	+12.796	−18.023	− 15	8	− 6	12	highly probable
46	63	+12.855	−21.798	− 22	12	+ 20	8	
47		+13.078	+58.800	− 21	10	+ 1	3	highly probable
48	48	+15.737	+ 1.195	+ 273	12	− 46	3	nonmember
49		+16.997	+10.491	− 9	15	− 18	3	highly probable
50		+28.929	+29.568	− 256	17	− 60	10	nonmember
51	24	+29.053	−22.296	− 13	19	− 9	7	highly probable
52		+29.853	+38.142	− 14	12	− 2	3	highly probable
53	16	+32.887	−32.958	− 26	7	+ 13	18	highly probable
54		+38.159	−56.076	+ 26	19	+ 39	14	
55		+44.951	+37.649	− 5	19	+ 26	14	
56		+48.308	−35.554	− 31	18	+ 4	15	
57	AB	+52.209	−10.640	+ 65	4	− 87	11	nonmember
58		+53.577	+21.982	− 15	15	− 14	13	highly probable
59	AE	+56.298	−18.878	− 13	13	+ 20	11	highly probable
60	S2	+56.630	−38.764	− 35	4	− 14	8	
61		+57.188	+80.228	+ 26	···	− 11	···	
62	AA	+61.226	−10.636	− 9	12	− 48	13	
63	AC	+64.716	− 8.007	+ 9	12	− 7	7	highly probable
64	AD	+68.420	−17.406	− 32	7	+ 53	7	
65	S4	+71.142	−52.900	− 9	9	− 14	2	highly probable
66		+71.795	−53.520	+ 20	30	+ 6	8	
67		+75.240	−43.070	+ 44	10	+ 7	9	
68		+79.595	+20.945	− 39	7	+ 19	17	
69		+86.036	+38.016	+ 25	15	+ 26	19	
70		+87.658	+52.570	+ 55	17	− 1	2	

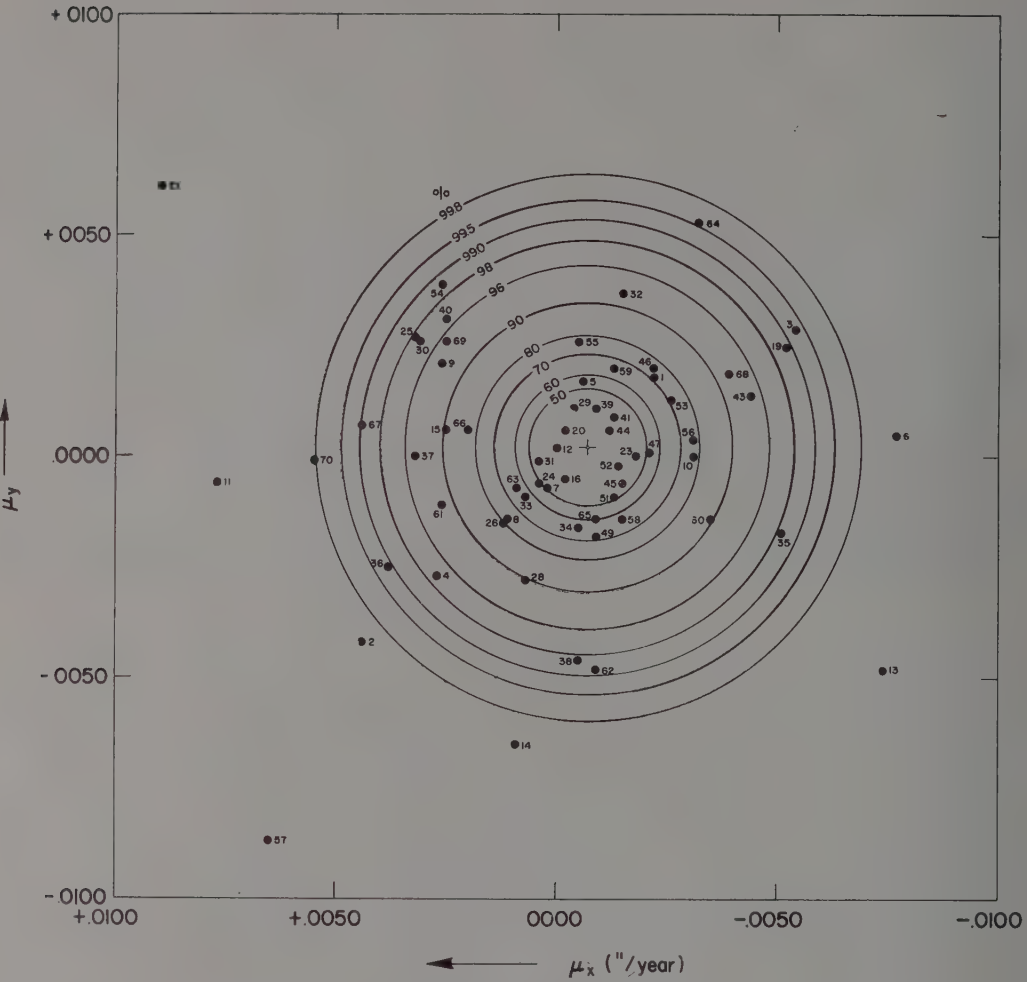

FIG. 2. Proper motion vector diagram of NGC 129.

are nearly identical within the limits of their accuracy. The standard deviations for groups B and C, however, are on average about 50% larger than those found for group A, located around the apparent cluster center. We may therefore conclude with some confidence that group A not only contains a considerably larger number of cluster members than groups B and C, but that it represents indeed essentially the nucleus of the cluster.

The mean relative motion $\bar{\mu}_x = -0''.0007/a$, $\bar{\mu}_y = +0''.0002/a$, obtained for the region of the apparent cluster center, can therefore be adopted as the mean motion of the cluster. The respective standard deviations, found to be $\pm 0''.0021/a$ and $\pm 0''.0022/a$, represent a combination of the true dispersion in the individual proper motions and of the errors with which these motions have been determined.

V. CLUSTER MEMBERSHIP

A proper motion vector diagram for the stars within $\pm 0''.0100/a$ in μ_x and μ_y of the adopted mean cluster motion is shown in Fig. 2, the mean relative motion of the cluster being indicated by a cross.

The small number of stars on the plates and the absence of an obvious separation between cluster and field stars complicates the selection of cluster members on the basis of their relative proper motions only. However, we have attempted to establish the probability of any star being a cluster member using a method suggested by S. D. C. Munday of the Royal Military College of Science, England.

In our discussion of the mean cluster motion the standard deviations of the relative motions for a group of stars likely to contain a large percentage of cluster members were found to be of the order of $\pm 0''.0020/a$.

This value for the standard deviation of the proper motion vectors in Fig. 2 from the mean cluster motion was adopted and it was assumed that these deviations follow a normal distribution. The radii R_i of a series of percentage zones P_i, containing various percentage areas of the normal distribution, were then computed. Circles of radius R_i were drawn in Fig. 2 and the stars contained within each circle were counted. These star counts n_i were then multiplied by a factor of $100/P_i$ to reduce them to unit area. We then have for each percentage zone the number of stars contained within it, reduced to unit area, equal to, or greater than, the total number of cluster members x_i, or $n_{i,r}=x_i+$number of field stars. The results of these star counts within a series of percentage circles in the proper motion vector diagram are given in Table III. Columns 1 and 2 give the percentage zones P_i and the corresponding proper motion vectors R_i, respectively. Column 3 contains the number of stars n_i counted within each percentage circle. The factors $100/P_i$ for reduction of the star counts n_i to unit area are listed in column 4. Column 5 contains the values $n_{i,r}$, the number of stars counted within each percentage circle, reduced to unit area.

The plot of P_i vs $n_{i,r}$, given in Table III, shows a rapid decrease of $n_{i,r}$ for the percentage zones $P=99.8\%$ to $P=80\%$, after which it remains reasonably constant at a mean value of 30. For $P_i<50\%$ the number of stars in the sample becomes too small to make an extension of the star counts beyond $P=50\%$ meaningful. The marked fall in $n_{i,r}$ derived from the percentage zones $99.8\%\geqslant P_i\geqslant 80\%$ clearly reflects the decreasing contribution of field stars to the total number of stars within the corresponding proper motion limits. The near constancy of $n_{i,r}$ obtained from the percentage zones $P_i\leqslant 75\%$ indicates that within $P=75\%$, corresponding to a vector length $0''.0023/a$ around the mean cluster motion in the proper motion vector diagram, the number of field stars becomes very small compared with the number of cluster members. On the basis of proper motions only we may therefore safely assume that virtually all of the 25 stars counted within the 75%-circle are members of the cluster. Of these 25 stars, for which the remark "highly probable" has been entered under "Membership" in the last column of Table II, 14 are contained within a circle of 10 minutes of arc radius in the central region of the cluster, while the remaining 11 stars are found at distances up to 20 minutes of arc or more from the apparent cluster center.

TABLE III. Distribution of proper motion vectors and probabilities of cluster membership.

P_i (1)	R_i (2)	n_i (3)	$100/P_i$ (4)	$n_{i,r}$ (5)	p_i (6)
99.8%	62.0	56	1.000	56	30:26
99.5	56.0	53.5	1.005	54	30:24
99.0	51.5	51.5	1.01	52	30:22
98.0	47.0	46	1.02	47	30:17
96.0	41.1	42	1.04	44	30:14
94.0	37.6	38	1.065	40	30:10
90.0	32.9	34	1.11	37	30:7
85.0	28.6	32	1.18	38	30:8
80.0	25.6	31	1.25	39	30:9
75.0	23.1	25	1.33	33	
70.0	20.8	23	1.43	33	
65.0	18.2	21	1.54	32	
60.0	16.8	17	1.67	28	
55.0	14.3	15	1.82	27	
50.0	13.5	14	2.00	28	

TABLE IV. Star counts within 20′ of the apparent cluster center.

Distance from cluster center	Number of stars brighter than $V=12.4$	Number of cluster members
20′.0–17′.3	16	0
17′.3–14′.2	20	4
14′.2–10′.0	20	4
within 10′	40	24

From the corrected star counts $n_{i,r}$, derived from the percentage zones $75\%\geqslant P_i\geqslant 50\%$, and from certain assumptions concerning the dispersion in the proper motions of the cluster stars, we obtain 30 as the approximate total number of cluster members brighter than $V=12.4$ in the region of NGC 129 covered by this investigation. Adopting this figure we can now find a measure of the probability of cluster membership for any star located within any percentage circle in the proper motion vector diagram. Its values for $99.8\%\geqslant P_i\geqslant 80.0\%$ are given in the last column of Table III. Stars located outside the 99.8%-circle in Fig. 2 can, on the basis of their proper motions, definitely be considered not to be members of the cluster and they have been marked "nonmember" in Table II. For all stars located between the 99.8%- and the 75%-circles a rapid estimate of the probability of cluster membership can be made by inspecting Fig. 2 and using the last column of Table III.

Since cluster center and plate center do not coincide on the first epoch plates, this proper motion survey does not extend as far to the south as it does to the north of the apparent cluster center. A correction factor of $\frac{4}{3}$ should therefore be applied to reduce the estimated total number of cluster members found for the area covered by this investigation to the entire field within 20 minutes of arc around the cluster center. We thus obtain 40 to be the total number of cluster members brighter than $V=12.4$.

An attempt was made to verify this estimate by making star counts in the region of the cluster center and its vicinity. Concentric circles of 10, 14.2, 17.3, and 20 minutes of arc radius were drawn around star No. 39, located in the adopted cluster center, on a photographic chart of the cluster region (Fig. 1). The three rings formed by these circles are of equal area and identical with that contained within the circle of 10 minutes of arc radius. Because of the incompleteness of the proper motion survey south of the cluster center,

TABLE V. Photometric data, spectral types, and probabilities for cluster membership of 16 stars in NGC 129.[a]

No. (1)	A (2)	V_0 (3)	$(B-V)_0$ (4)	E_{B-V} (5)	Sp. (6)	Probability (7)
30	125	10.23	−0.14	0.53		30:17
32	121	10.37	−0.14	0.59		30:10
34	DL Cas[b]	7.38	+0.71	0.53		highly probable
36	115	8.62	−0.15	0.94		30:24
38	A	7.34	+0.45	0.50	F5 Ib	30:22
39	105	9.70	−0.15	0.50		highly probable
41	96	10.58	−0.15	0.59		highly probable
42	B	9.30	+0.41	0.00	F6 V	nonmember
43	71	10.79	−0.12	0.51		30:14
44	C	9.17	−0.14	0.57	B6 III–IV	highly probable
45	61	10.00	−0.18	0.58		highly probable
46	63	10.14	−0.15	0.58		30:9
51	24	9.88	−0.17	0.62		highly probable
53	16	11.00	−0.09	0.46		highly probable
57	AB	8.08	+1.55	0.21	M0 IIp	nonmember
62	AA	6.29	+1.16	0.76	K2 Ib	30:22

[a] Explanation of column headings:
Col. (1): star numbers.
Col. (2): star numbers by Arp, Sandage, and Stephens (1959).
Col. (3), (4), and (5): photometric data.
Col. (6): spectral types.
Col. (7): probabilities of cluster membership as defined in Sec. V.

[b] Photometric data are mean values.

all stars brighter than $V=12.4$ were counted only in the northern halves of these three rings and of the area within the 10 minute of arc circle. If we multiply the counted numbers of stars by a factor 2 to reduce them to concentric zones of the entire field around the cluster center, we obtain the numbers given in the second column of Table IV.

Since from the proper motions no stars with high probability of cluster membership have been found in the zone between 17'.3 and 20'.0 from the apparen cluster center, we may consider all 16 stars counted i this zone to be field stars. If we assume uniform distribution of the field stars over the entire cluster region we obtain the number of cluster stars for each zon given in the third column of Table IV. From star count we thus find the total number of cluster members located within 20 minutes of arc from the cluster center and brighter than $V=12.4$, to be 32, in good agreemen

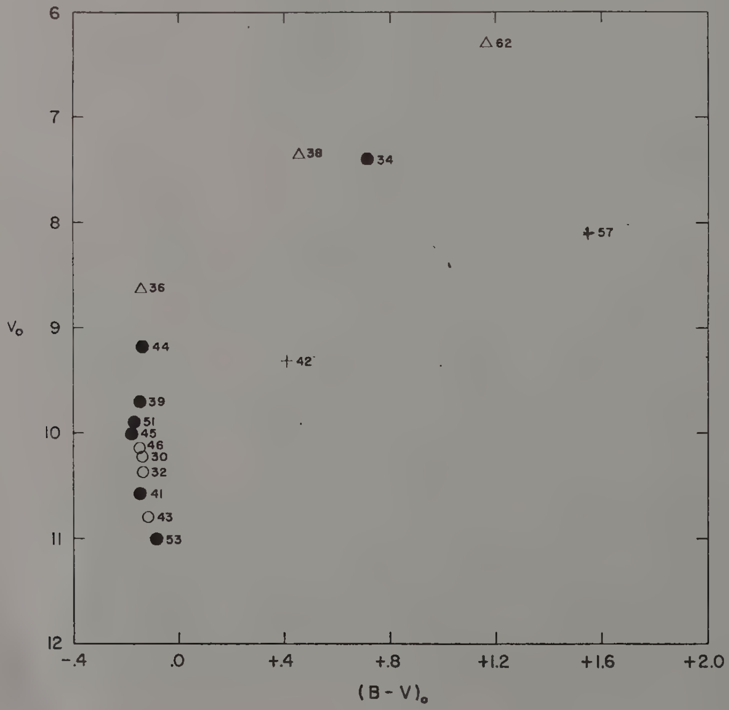

FIG. 3. Color-magnitude diagram for the 16 stars of Table V. Filled circles ● represent "highly probable" cluster members; open circles ○ are used for stars with a probability of cluster membership between 4:1 and 2:1; triangles △ are used for stars with a probability of cluster membership of about 1:1; crosses + represent "nonmembers."

vith the total of 40 derived from the observed relative roper motions.

A further check of the validity of the method used o distinguish cluster members from field stars is possible by comparing our results with the photometric lata obtained by Arp, Sandage, and Stephens (1959). There are 14 stars brighter than $V \doteq 12.4$ within their egion 1 for which relative proper motions have been letermined. The photometric data and the available spectral types, as well as indications of the probabilities f being cluster members, for these 14 stars and for the wo supergiants Nos. 57 and 62, located about 10 ninutes of arc east of the cluster center, are listed in Table V. Individual reddening and absorption corections were applied to the colors and magnitudes neasured by Arp *et al.*, using for most of the stars the omogram by Johnson (1958) and $A_V = 3.0\, E_{B-V}$. For he stars of known spectral type the intrinsic colors were aken from Johnson and Morgan (1953). For star No. 34 = DL Cas the corrected mean color and magnitude vas adopted from Table 5 of Arp, Sandage, and Stephens (1959). The color–magnitude diagram for the 16 stars f Table V is given in Fig. 3. Inspection of both Table V nd Fig. 3 shows immediately that all stars which have een clearly recognized as cluster members on the basis f photometry have also been designated as "highly robable" members or have at least a probability of eing cluster members of better than 2:1, based on heir relative proper motions only. On the other hand, stars defined as nonmembers or stars with low probability of being cluster members, based on their proper motions, are either clearly identified as nonmembers by means of photometric observations, or their positions in the color-magnitude diagram alone do not permit a lefinite decision in favor of or against their being members of the cluster. We may therefore conclude that the method of distinguishing cluster members from field stars on the basis of their relative proper motions alone has a satisfactory degree of reliability.

VI. SUPERGIANTS IN THE REGION OF NGC 129

It is of great interest to note that on the basis of its roper motion the Cepheid DL Cas = No. 34 has a very high probability of being a member of the cluster in accordance with previous suggestions by Van den Bergh (1957), based on its apparent proximity to the cluster center, and by Kraft (1958), inferred from spectroscopic and photometric observations.

For the supergiants Nos. 38 and 62, which have almost identical proper motion, the probability of being cluster members is only 1.4:1. Star No. 38, located near the apparent cluster center, has a proper motion of only +0.″0002/year in right ascension, relative to the mean cluster motion, while its proper motion in declination relative to the cluster is −0.″0050/year. Only one million years ago the star would therefore have been located more than one degree north of the apparent center of the cluster.

Star No. 57, located near star No. 62, approximately 10 minutes of arc east of the cluster center, can definitely be considered not to be a member of the cluster on account of its large relative motion. The direction of the proper motions of star Nos. 57 and 62 also rules out the possibility that they may have moved from the vicinity of the cluster center to their present locations.

It therefore appears that of the four supergiants located in the region of NGC 129 the Cepheid DL Cas is the only one which has a high probability of being a physical member of the cluster.

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Observations and Times of Minima for Three W Ursae Majoris Systems

Beverly B. Bookmyer
Flower and Cook Observatory, University of Pennsylvania, Philadelphia, Pennsylvania

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Times of minimum light determined from photoelectric observations are given for AK Herculis, W Ursae Majoris and AM Leonis. New light elements have been derived for AM Leonis. Earlier discussions of AK Herculis have been on the basis of partial eclipses. The present observations, however, indicate an interval of constant light at the bottom of primary minimum of approximately 35 minutes' duration.

1. OBSERVATIONS

AK HERCULIS was observed photoelectrically in two wavelength bands with the 28-inch reflecting telescope of Flower and Cook Observatory. The output of the 1P21 multiplier photocell used was fed into a direct current amplifier and this amplified current was registered by a Brown recorder. The combination of the photocell and a yellow filter, Corning 3384 with a thickness of 3.0 mm, gave an effective wavelength of approximately 5300 A, and the combination of the photocell and two filters, a Corning 5543 with a thickness of 2.5 mm and a Corning 3060 with a thickness of 2.0 mm, gave an effective wavelength of approximately 4420 A. These effective wavelengths were determined from the combination of the response curve of the photocell, obtained from data published by the manufacturer, and the sensitivity curves of the individual filters, measured by Binnendijk.

The equipment used for the observations of W Ursae Majoris and AM Leonis was the Newton Lacy Pierce photoelectric photometer attached to the 15-inch horizontal refracting telescope which has a 25-inch siderostat. A detailed description of this photometer and its operation has been given by Blitzstein (1956). A multiplier photocell 1P21 and a yellow filter, Corning 3384 with a thickness of 1.75 mm, were employed for all the observations. The effective wavelength of this photocell-filter combination is approximately 5300 A. In addition, for the 1960 observations of W UMa a combination of two filters was used, a Corning 5113 with a thickness of 1.0 mm and a Corning 3060 with a thickness of 0.75 mm, giving an effective wavelength of approximately 4250 A to the photocell-filter combination. These effective wavelengths were determined from the combination of the response curve of the photocell, measured by H. Fliegel, and the sensitivity curves of the individual filters, obtained from data published by the manufacturer.

Table I. Variable and comparison stars.

Star	BD No.	R.A. (1900)	Dec. (1900)	Sp.
W UMa	56° 1400	$9^h\ 36^m\ 44^s$	56° $24'.5$	F8p+F8p
Comp. star	56 1399	9 35 40	56 06.8	K0
AK Her	16° 3130	$17^h\ 09^m\ 29^s$	16° $28'.1$	F2+F6
Comp. star	16 3123	17 07 59	16 37.1	F8
Check star	16 3124	17 08 12	16 20.6	G0
AM Leo	10° 2234	$10^h\ 56^m\ 58^s$	10° $26'.1$	F8
Comp. star	10 2235	10 56 58	10 16.1	· ·

Table I contains data concerning the variable and comparison stars. Table II lists the times of the individual observations of the three systems in heliocentric Julian Days and decimals of a day along with the derived differences, variable minus comparison star. Although each of the systems observed is one component of a visual double star, it was unnecessary to free the observations from the light of the respective visual companions since the observations were not being utilized for the derivation of orbital elements. It was also unnecessary to apply a correction for differential extinction because of the proximity, in each instance, of the comparison star to the variable.

2. AK HERCULIS

In 1917 Metcalf (Pickering 1917) discovered the variability of the eclipsing system AK Her from Harvard plates. Observations since that time have revealed asymmetry in the light curve and there may be variations in the light curve from cycle to cycle. A recent investigation by Schmidt and Herczeg (1959) indicated that primary minimum was deeper by 0.150 mag. than secondary and that primary maximum was somewhat higher than secondary in both 1957 and 1958. They found also that the light curve was asymmetric with secondary minimum arriving later than the half-period point.

Orbital determinations of this system have been based on the assumption that the minima are caused by partial eclipses. The observations of the present study, however, indicated a flat primary minimum with the duration of the stationary interval of minimum light lasting approximately 35 minutes. Since more reliable information can be obtained from systems which undergo complete eclipse than from those whose eclipses are partial, additional observations of AK Her were made by Binnendijk and Bookmyer during the summer of 1960. A well-covered light curve was obtained in two wavelength regions and these additional observations confirmed the fact of complete eclipses, with an annular eclipse occurring at primary minimum. The orbital

TABLE II. Observations.

JD ⊙ 436000+	V–C	JD ⊙ 2436000+	V–C	JD ⊙ 2436000+	V–C
AK Herculis, Yellow					
754.6670	$-0^{m}.239$	754.6897	$-0^{m}.058$	754.7128	$-0^{m}.017$
.6684	0.232	.6911	−0.034	.7198	0.012
.6698	0.222	.6966	+0.005	.7223	0.012
.6712	0.207	.6980	0.004	.7250	0.025
.6772	0.175	.6994	0.002	.7274	0.050
.6786	0.160	.7008	0.002	.7295	0.065
.6800	0.155	.7073	+0.000	.7384	0.157
.6813	0.135	.7087	−0.012	.7406	0.200
.6869	0.082	.7107	−0.012	.7420	−0.194
.6883	−0.069				
AK Herculis, Blue					
754.6677	$-0^{m}.207$	754.6904	$-0^{m}.022$	754.7212	$+0^{m}.042$
.6691	0.201	.6918	−0.007	.7237	+0.003
.6705	0.191	.6973	+0.041	.7267	−0.029
.6719	0.170	.6987	0.038	.7281	0.041
.6779	0.139	.7001	0.032	.7302	0.060
.6793	0.125	.7015	0.029	.7377	0.131
.6807	0.120	.7080	0.042	.7391	0.144
.6821	0.095	.7101	0.041	.7413	0.176
.6876	0.039	.7114	0.038	.7434	−0.195
.6890	−0.036	.7135	+0.035		
AM Leonis, Yellow					
988.6536	$-0^{m}.433$	988.7031	$-0^{m}.184$	995.6360	$-0^{m}.114$
.6576	0.393	.7071	0.182	.6405	0.099
.6616	0.355	.7111	0.250	.6455	0.121
.6661	0.345	.7175	0.293	.6495	0.129
.6701	0.248	.7276	0.408	.6535	0.166
.6741	0.202	.7351	0.480	.6575	0.217
.6781	0.183	.7406	0.520	.6615	0.278
.6821	0.103	995.6160	0.337	.6655	0.306
.6861	0.110	.6200	0.294	.6695	0.367
.6901	0.072	.6240	0.212	.6735	0.389
.6946	0.097	.6280	0.190	.6775	−0.399
.6986	−0.098	.6320	−0.138		

JD ⊙ 2436000+	V–C	JD ⊙ 2436000+	V–C	JD ⊙ 2436000+	V–C
W Ursae Majoris, Yellow					
634.6810	$-1^{m}.103$	634.7812	$-1^{m}.050$	635.7611	$-0^{m}.823$
.6846	1.090	.7854	1.080	.7647	0.866
.6891	1.082	635.6819	1.108	.7677	0.919
.6990	1.007	.6849	1.108	.7713	0.962
.7020	0.979	.6885	1.079	.7743	1.002
.7062	0.923	.6915	1.059	.7779	1.041
.7098	0.872	.6957	1.045	.7809	1.044
.7143	0.810	.6987	1.022	.7845	1.071
.7176	0.750	.7023	0.986	.7875	1.068
.7224	0.663	.7065	0.934	.7911	1.099
.7266	0.581	.7101	0.895	968.6532	1.123
.7308	0.528	.7131	0.840	.6612	1.082
.7338	0.528	.7185	0.758	.6688	1.056
.7389	0.522	.7215	0.704	.6764	0.985
.7434	0.533	.7251	0.630	.6844	0.849
.7476	0.582	.7281	0.561	.6916	0.706
.7506	0.658	.7317	0.505	.6988	0.516
.7548	0.741	.7347	0.519	.7060	0.449
.7578	0.787	.7383	0.535	.7136	0.502
.7614	0.841	.7413	0.522	.7208	0.567
.7644	0.891	.7449	0.540	.7284	0.704
.7680	0.943	.7479	0.580	.7368	0.885
.7710	0.979	.7515	0.642	.7452	1.003
.7746	1.004	.7545	0.695	.7596	1.058
.7776	−1.042	.7581	−0.801	.7696	−1.103
W Ursae Majoris, Blue					
968.6524	$-1^{m}.377$	968.6908	$-0^{m}.886$	968.7276	$-0^{m}.875$
.6604	1.269	.6980	0.709	.7360	1.118
.6680	1.276	.7052	0.617	.7444	1.208
.6756	1.148	.7128	0.663	.7608	1.296
.6836	−1.022	.7200	−0.727	.7688	−1.351

elements have been derived by Binnendijk and appear in the following article.

One time of minimum light was determined from the 1959 observations and is given in heliocentric Julian Days,

$$\text{Min. I} = \text{JD Hel. } 2436754.7085.$$

The published times of minimum light have been tabulated by Woodward (1942) and Schmidt and Herczeg (1959). The $(O–C)$ values for all these times of minimum were obtained by using the light elements

$$\text{Min. I} = \text{JD Hel. } 2422977.254 + 0^{d}.42152207\, E.$$

These elements yield an $(O–C)$ value of $+0^{d}.0056$ for the time of minimum determined from the present study.

From a plot of the mean values of these residuals versus time, it can be seen that the period of AK Her decreased continuously from the time of the earliest observations on Harvard plates until about 1940. Then the variation changed sign and the period increased considerably and is now continuing to increase.

3. W URSAE MAJORIS

This short-period eclipsing binary has been observed extensively since its discovery by Müller and Kempf (1903). It shows variations in its light curve, marked period irregularities, and peculiarities in its spectra. A complete compilation of the literature up until 1942 has been given by Woodward (1942), and the later data have been summarized by Schmidt and Schrick (1956). Plaut (1950, 1953) has given the orbital elements of the system derived from the photoelectric observations of Huffer (1934), but the most recent orbital determination is that made by Kwee (1956) from a light curve which he observed photoelectrically without a filter in one night. He based his computation on primary minimum's being caused by a total grazing eclipse, and he was forced to assume extra light in the amount of 25% of the total light of the system.

The present observations were made on two successive nights in 1959 and on one night in 1960. On each night a primary minimum was observed. The observations of the three nights are in satisfactory agreement except for four observations near minimum light in 1960. These differ systematically by several hundredths

Table III. Times of minimum light, AM Leonis.

JD ⊙ 2430000+	Min.	Meth.	*O–C*	Reference
5569.934[a]	I			
5570.854	II	pe	$0^{d}.000$	Worley and Eggen (1956)
5572.683	II	pe	0 .000	Worley and Eggen (1956)
5593.718	I	pe	+0 .001	Worley and Eggen (1956)
5594.815	I	pe	+0 .001	Worley and Eggen (1956)
5607.799	II	pe	−0 .001	Worley and Eggen (1956)
5900.437	II	pe	−0 .002	Abrami (1959)
5901.355	I	pe	+0 .001	Abrami (1959)
5901.536	II	pe	−0 .001	Abrami (1959)
5924.393	I	v	−0 .004	Rudolph (1960)
5924.397	I	v		Rudolph (1960)
5933.549	I	v	+0 .005	Rudolph (1960)
5933.549	I	v		Rudolph (1960)
5934.461	II	v	+0 .002	Rudolph (1960)
5951.463	I	v	−0 .002	Rudolph (1960)
5951.467	I	v		Rudolph (1960)
5951.467	I	v		Rudolph (1960)
5951.469	I	pe	+0 .001	Abrami (1959)
5952.383	II	pe	0 .000	Abrami (1959)
6702.455	I	pe	+0 .001	Abrami (1959)
6713.428	I	pe	0 .000	Abrami (1959)
6988.6917	II	pe	0 .0000	Bookmyer
6995.6413	II	pe	−0 .0006	Bookmyer

[a] Min. extrapolated by Abrami.

of a magnitude from the 1959 observations. There is a suggestion of a brief stationary interval of light intensity at minimum. The times of minimum light given in heliocentric Julian Days are

$$\text{Min. I} = \text{JD Hel. } 2436634.7367$$

$$\text{Min. I} = \text{JD Hel. } 2436635.7377$$

$$\text{Min. I} = \text{JD Hel. } 2436968.7090.$$

After decreasing slowly for about 40 years, the period showed a change in sign about 1950 and is now increasing. Chou (1959) made a recent period study. His observed epoch of minimum and suggested period

$$\text{Min. I} = \text{JD Hel. } 2436611.7163 + 0^{d}.3336384\,E$$

yield (*O–C*) values of $-0^{d}.0006$ for each of the 1959 observations and $-0^{d}.0004$ for the 1960 observations.

4. AM LEONIS

The variability of AM Leo was discovered by Hoffmeister (1935) and was found to be a system of the W UMa type by Worley and Eggen (1956). Abrami (1959) observed the system photoelectrically without a filter. He made an orbital determination based on complete eclipse, with total eclipse occurring at time of primary minimum. The difference in magnitude between the components of the visual double was not measured directly but was included as an unknown in the orbital computation.

From their photoelectric observations Worley and Eggen determined the light elements of AM Leonis. According to Abrami's observations, however, their epoch corresponds to a secondary minimum. He redetermined the light elements to be

$$\text{Min. I} = \text{JD Hel. } 2435569.939 + 0^{d}.365800\,E.$$

The present observations were made on two nights in 1960, and a secondary minimum was observed on each night. The combined light of the visual components of the variable star was measured, as was the total light from both components of the comparison star. The times of minimum light given in heliocentric Julian Days are

$$\text{Min. II} = \text{JD Hel. } 2436988.6917$$

$$\text{Min. II} = \text{JD Hel. } 2436995.6413.$$

The observations indicate a period shorter than that determined by Abrami. Since there has been no evidence of a period change or of a displacement of secondary minimum, the method of least squares was employed to determine the light elements of the system. Only the published photoelectric times of minimum shown in Table III were utilized in the computation. The new light elements are

$$\begin{array}{l} \text{Min. I} = \text{JD Hel. } 2435593.7168 + 0^{d}.36579912\,E \\ \qquad\qquad\qquad\qquad\qquad \pm \quad 2 \pm \qquad\qquad 12\ \text{(p.e.)}. \end{array}$$

Using these, (*O–C*) values were found for all the published times of minimum light and are shown in Table III. It is not expected that this period will hold for any length of time since all the known W UMa systems have shown period variations when observed over a long interval of time.

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The Light Variation and Orbital Elements of AK Herculis

L. Binnendijk
Flower and Cook Observatory, University of Pennsylvania, Philadelphia, Pennsylvania

Photoelectric observations are presented of 520 yellow and 518 blue magnitudes of AK Herculis made during one night in 1959 and ten nights in 1960. The minima show constant light during an interval of 35 minutes. The orbital elements were derived for complete eclipses; the primary minimum is caused by an annular eclipse.

1. INTRODUCTION

THE eclipsing variable AK Herculis is the brighter component of the double star ADS 10408. The ainter component is at a distance of 4″.7 in position ngle 322° and is four magnitudes fainter than the ariable at maximum light. The variability of this W Jrsae Majoris type system was discovered by Metcalf Pickering 1917). Photographic light curves were ublished by Jordan (1929) and Woodward (1942); hotoelectric light curves were observed by Stebbins see Woodward), Seyfert and Mason (1951), Labs and tock (1953), and Schmidt and Herczeg (1959). Spec-rographic observations were made by Sanford (1934) vho observed the absorption lines of the brighter omponent only.

2. OBSERVATIONS

In the present study the variable was observed photo-lectrically in two wavelength regions with the 28-inch eflecting telescope of the Flower and Cook Observatory. yellow filter, Corning 3384 of thickness 3.0 mm was sed with an effective wavelength of 5300 A for the ell-filter combination. The combination of two filters, orning 5543 of thickness 2.5 mm and Corning 3060 of hickness 2.0 mm, was used for the blue filter, with an ffective wavelength of 4420 A for the cell-filter com-ination. The output of a 1P21 multiplier photocell was ed into a direct current amplifier and the amplified urrent was registered by a Brown recorder. The dura-ion of each deflection was one minute.

Table I lists the BD numbers, the positions of the tars and the spectral types. The latter for the variable tar have been published by Roman (1956) who esti-nated the spectral types of both components from the omposite spectrum. The difference in magnitude etween the comparison star and check star was found o be constant. The extinction for each night was deter-nined from the observations of the comparison star, nd the correction for differential extinction was applied to the differences in magnitude between the variable star and the comparison star.

TABLE I. Variable and comparison stars.

Star	BD No.	R.A. (1900)	Dec. (1900)	Sp.
K Herculis	+16°3130	$17^h\ 9^m\ 29^s$	+16° 28′.1	F2+F6
omparison	16 3123	17 7 59	+16 37.1	F8
heck star	16 3124	17 8 12	+16 20.6	G0

TABLE II. Times of minima of AK Herculis.

JD ⊙	Min.	*O–C*	Meth.	Obs.[a]	Ref.
2423100.340	I	+0.002	v	Es	1
3573.502	I	+0.005	pg	Jo	2
3573.703	II	−0.055	pg	Jo	2
4680.626	I	+0.001	v	Sb	3
4949.555	I	−0.001	v	Ko	4
5413.227	I	−0.003	v	Sa	5
5442.738	I	+0.001	v	ML	6
5766.668	II	−0.008	pg	Jo	2
6141.195	I	−0.004	v	Wa	7
6894.449	I	−0.010	pe	St	8
7980.504	II	−0.006	pg	Zo	9
8329.7288	I	−0.012	pg	Ba	10
8429.9428	II	−0.009	pg	Ba	10
9486.601	II	−0.008	pg	Wo	8
9486.805	I	−0.015	pg	Wo	8
2430169.465	II	−0.009	pg	Wo	8
0169.673	I	−0.012	pg	Wo	8
0914.500	I	−0.015	v	Te	11
0934.524	II	−0.013	v	Te	11
2352.525	II	−0.012	v	Fr	11
2362.430	I	−0.013	v	Fr	11
3515.718	I	−0.0093	pe	S, M	12
3731.5405	I	−0.0065	v	D, P	13
3835.447	II	−0.005	v	D, P	13
4153.4827	I	−0.0075	pe	L, S	14
4213.760	I	−0.008	v	As	15
4253.384	I	−0.007	v	Po	16
4549.505	II	−0.005	v	Po	16
4813.59075	I	−0.0030	pe	Kw	17
4859.53645	I	−0.0032	pe	Kw	17
5960.5588	I	+0.0035	pe	S, H	18
6018.5197	II	+0.0051	pe	S, H	18
6025.4708	I	+0.0011	pe	S, H	18
6404.4205	I	+0.0025	pe	S, H	18
6405.4791	II	+0.0073	pe	S, H	18
6754.7085	I	+0.0056	pe	Bo	19
6757.6601	I	+0.0066	pe	Bi	..
7102.6774	II	+0.0081	pe	Bi	..
7111.7411	I	+0.0091	pe	Bi	..

[a] Es = Esch 1 (1924, 1937); Jo = Jordan 2 (1929); Sb = Sternberk 3 (1927); Ko = Kordylewski 4 (1927); Sa = Sacharow 5 (1953); ML = McLaughlin 6 (1929); Wa = Wasiutynski 7 (1931); St = Stebbins 8; Zo = Zonn 9 (1938); Ba = Bailey 10 (1941); Wo = Woodward 8 (1942); Te = Tedeschini 11; Fr = Fresa 11 (1950); S,M = Seyfert and Mason 12 (1951); D,P = Domke and Pohl 13, (1952); L,S = Labs and Stock 14 (1953); As = Ashbrook 15 (1952); Po = Pohl 16 (1955); Kw = Kwee 17 (1958); S,H = Schmidt and Herczeg 18 (1959); Bo = Bookmyer 19 (1961); Bi = Binnendijk.

3. TIMES OF MINIMA

A total of 520 yellow and 518 blue observations were made during one night in 1959 and ten nights in 1960. Three times of minima light were determined, using the method of Hertzsprung (1928). These are shown in Table II along with older data. Since the epoch pub-

TABLE IIIa. Yellow observations of AK Herculis.

JD Hel. 2430000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm
6757.6051	0.8694	−0.307	7058.7247	0.2270	−0.424	7104.7235	0.3518	−0.344	7111.8126	0.1696	−0.36
.6065	0.8727	0.302	.7261	0.2303	0.426	.7250	0.3554	0.349	.8141	0.1731	0.37
.6079	0.8760	0.298	.7361	0.2542	0.408	.7264	0.3587	0.345	.8154	0.1761	0.37
.6092	0.8793	0.286	.7376	0.2577	0.409	.7499	0.4144	0.231	.8168	0.1794	0.38
.6217	0.9089	0.219	.7390	0.2610	0.413	.7514	0.4180	0.215	7112.5925	0.0196	+0.03
.6231	0.9122	0.214	.7444	0.2737	0.412	.7534	0.4227	0.200	.5936	0.0223	0.02
.6245	0.9155	0.204	.7458	0.2772	0.407	7111.6034	0.6731	−0.372	.5953	0.0265	0.01
.6259	0.9188	0.187	.7474	0.2808	0.399	.6050	0.6770	0.386	.5968	0.0300	0.021
.6315	0.9320	0.126	.7526	0.2931	0.392	.6068	0.6814	0.389	.5982	0.0333	0.01
.6333	0.9364	0.114	.7542	0.2970	0.393	.6186	0.7094	0.398	.5995	0.0363	0.001
.6347	0.9397	0.105	7102.6090	0.3354	−0.355	.6205	0.7138	0.405	.6009	0.0396	0.01
.6369	0.9449	0.085	.6105	0.3390	0.374	.6219	0.7171	0.397	.6063	0.0525	−0.057
.6426	0.9584	0.034	.6119	0.3423	0.367	.6277	0.7308	0.400	.6078	0.0561	0.074
.6440	0.9617	0.028	.6237	0.3703	0.330	.6292	0.7344	0.409	.6099	0.0611	0.10
.6454	0.9649	+0.006	.6251	0.3735	0.333	.6304	0.7374	0.406	.6113	0.0643	0.11
.6536	0.9844	0.027	.6263	0.3766	0.319	.6318	0.7407	0.411	.6172	0.0784	0.171
.6551	0.9880	0.011	.6326	0.3914	0.266	.6383	0.7561	0.405	.6186	0.0816	0.191
.6565	0.9913	0.015	.6340	0.3947	0.259	.6397	0.7594	0.409	.6200	0.0850	0.204
.6620	0.0045	0.015	.6354	0.3980	0.264	.6411	0.7626	0.411	.6222	0.0902	0.20
.6634	0.0078	0.015	.6445	0.4197	0.217	.6425	0.7659	0.408	.6273	0.1022	0.242
.6648	0.0111	0.044	.6459	0.4230	0.203	.6832	0.8626	0.322	.6287	0.1055	0.251
.6664	0.0149	0.035	.6472	0.4260	0.203	.6888	0.8758	0.278	.6300	0.1086	0.272
.6731	0.0309	0.021	.6528	0.4395	0.168	.6909	0.8807	0.274	.6312	0.1116	0.277
.6752	0.0358	−0.002	.6542	0.4427	0.159	.6925	0.8846	0.265	.6367	0.1245	0.31
.6768	0.0397	0.020	.6556	0.4460	0.148	.6939	0.8878	0.261	.6381	0.1278	0.322
.6782	0.0429	0.028	.6613	0.4595	0.114	.7001	0.9027	0.229	.6395	0.1311	0.32
.6836	0.0556	0.073	.6628	0.4631	0.106	.7015	0.9060	0.215	.6408	0.1344	0.33
.6848	0.0586	0.081	.6641	0.4661	0.092	.7029	0.9093	0.208	.6464	0.1475	0.341
.6863	0.0622	0.120	.6700	0.4801	0.063	.7043	0.9126	0.199	.6478	0.1508	0.34
.6875	0.0649	0.124	.6716	0.4839	0.084	.7103	0.9268	0.160	.6492	0.1541	0.34
.6891	0.0688	0.149	.6731	0.4875	0.076	.7117	0.9301	0.128	.6506	0.1574	0.35
.6947	0.0819	0.171	.6784	0.5001	0.063	.7131	0.9334	0.127	.6559	0.1700	0.372
.6959	0.0850	0.190	.6799	0.5037	0.082	.7145	0.9367	0.097	.6573	0.1733	0.372
.6974	0.0885	0.206	.6813	0.5070	0.073	.7199	0.9496	0.054	.6587	0.1766	0.372
.6995	0.0935	0.207	.6966	0.5432	0.120	.7214	0.9532	0.043	.6602	0.1802	0.38
.7009	0.0968	0.223	.6980	0.5465	0.131	.7228	0.9565	0.031	.6709	0.2057	0.41
.7065	0.1099	0.277	.6994	0.5498	0.145	.7242	0.9598	0.022	.6723	0.2091	0.41
.7079	0.1132	0.287	.7049	0.5630	0.184	.7256	0.9631	0.003	.6737	0.2123	0.42
.7095	0.1171	0.294	.7063	0.5663	0.199	.7270	0.9664	0.005	.6751	0.2156	0.42
.7112	0.1212	0.318	.7077	0.5696	0.204	.7320	0.9782	+0.031	.6810	0.2296	0.41
.7169	0.1347	0.321	.7133	0.5828	0.237	.7332	0.9812	0.025	.6827	0.2337	0.41
.7183	0.1380	0.331	.7146	0.5861	0.258	.7346	0.9845	0.028	.6840	0.2368	0.41
.7197	0.1412	0.337	.7161	0.5896	0.267	.7360	0.9878	0.028	.6855	0.2403	0.41
.7211	0.1445	0.337	.7222	0.6039	0.274	.7374	0.9911	0.028	.6908	0.2530	0.42
.7266	0.1577	0.361	.7237	0.6075	0.298	.7387	0.9941	0.026	.6921	0.2560	0.42
.7280	0.1610	0.367	.7251	0.6108	0.316	.7402	0.9977	0.023	.6936	0.2596	0.42
.7295	0.1646	0.373	.7306	0.6240	0.323	.7450	0.0092	0.040	.6951	0.2631	0.422
.7316	0.1695	0.365	.7320	0.6273	0.349	.7465	0.0128	0.025	.7001	0.2750	0.42
7058.6424	0.0318	+0.005	.7334	0.6305	0.329	.7478	0.0158	0.027	.7015	0.2783	0.42
.6438	0.0351	0.001	7104.6245	0.1168	−0.274	.7492	0.0191	0.025	.7029	0.2815	0.41
.6454	0.0389	−0.005	.6285	0.1264	0.290	.7506	0.0224	0.025	.7043	0.2848	0.422
.6512	0.0526	0.044	.6362	0.1445	0.341	.7520	0.0257	0.020	.7151	0.3106	0.39
.6529	0.0567	0.072	.6380	0.1489	0.335	.7566	0.0367	0.007	.7165	0.3139	0.41
.6543	0.0600	0.093	.6400	0.1536	0.347	.7580	0.0400	0.012	.7179	0.3172	0.404
.6641	0.0834	0.186	.6486	0.1742	0.357	.7595	0.0435	−0.029	.7193	0.3205	0.39
.6659	0.0875	0.197	.6508	0.1794	0.374	.7610	0.0471	0.046	.7249	0.3337	0.364
.6674	0.0911	0.218	.6523	0.1830	0.383	.7624	0.0504	0.060	.7262	0.3370	0.37
.6729	0.1043	0.273	.6673	0.2184	0.403	.7686	0.0652	0.117	.7276	0.3403	0.372
.6744	0.1078	0.277	.6690	0.2225	0.404	.7701	0.0688	0.128	.7290	0.3436	0.377
.6758	0.1111	0.287	.6710	0.2272	0.423	.7714	0.0718	0.140	.7346	0.3568	0.361
.6866	0.1366	0.313	.6768	0.2409	0.422	.7729	0.0754	0.155	.7360	0.3601	0.35
.6883	0.1408	0.327	.6785	0.2450	0.417	.7792	0.0902	0.204	.7374	0.3634	0.351
.6897	0.1440	0.332	.6799	0.2483	0.427	.7814	0.0954	0.225	.7387	0.3666	0.34
.6947	0.1559	0.340	.6817	0.2527	0.417	.7828	0.0987	0.234	.7450	0.3815	0.30
.6963	0.1597	0.346	.6894	0.2708	0.402	.7841	0.1020	0.245	.7470	0.3861	0.307
.6978	0.1633	0.358	.6912	0.2752	0.407	.7903	0.1166	0.285	.7485	0.3897	0.29
.7064	0.1836	0.378	.6973	0.2895	0.397	.7922	0.1212	0.291	.7499	0.3930	0.294
.7078	0.1869	0.383	.6992	0.2942	0.390	.7936	0.1245	0.293	.7512	0.3963	0.284
.7092	0.1902	0.376	.7007	0.2978	0.392	.7950	0.1278	0.304	.7572	0.4103	0.254
.7140	0.2017	0.397	.7027	0.3024	0.390	.8022	0.1449	0.331	.7583	0.4131	0.23
.7154	0.2050	0.404	.7142	0.3296	0.370	.8041	0.1492	0.345	.7597	0.4164	0.21
.7168	0.2083	0.406	.7155	0.3329	0.371	.8055	0.1525	0.349	.7651	0.4293	0.191
.7232	0.2234	0.420	.7169	0.3362	0.364	.8070	0.1561	0.356	.7665	0.4325	0.17

TABLE IIIa. (Continued).

Hel. 0000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm	JD Hel 2430000+	Phase	Δm
2.7679	0.4358	−0.167	7113.6954	0.6362	−0.334	7114.6387	0.8740	−0.301	7136.7309	0.2841	−0.416
.7692	0.4389	0.154	.7039	0.6563	0.346	.6401	0.8773	0.277	.7322	0.2874	0.414
.7756	0.4540	0.116	.7054	0.6599	0.354	.6415	0.8805	0.288	.7391	0.3036	0.408
.7769	0.4573	0.103	.7067	0.6629	0.350	.6429	0.8838	0.261	.7406	0.3071	0.409
.7783	0.5606	0.095	.7082	0.6664	0.353	.6486	0.8976	0.235	.7420	0.3104	0.411
.7797	0.4638	0.087	.7133	0.6785	0.368	.6500	0.9009	0.222	.7434	0.3137	0.407
.7849	0.4762	0.060	.7148	0.6821	0.382	.6514	0.9042	0.215	.7489	0.3269	0.401
.7867	0.4803	0.074	.7163	0.6857	0.375	.6528	0.9075	0.204	.7502	0.3299	0.396
.7881	0.4836	0.078	.7179	0.6895	0.387	.6588	0.9217	0.151	.7517	0.3335	0.390
.7894	0.4869	0.070	.7235	0.7027	0.397	.6608	0.9264	0.147	.7538	0.3384	0.384
.7951	0.5004	0.087	.7248	0.7060	0.379	.6623	0.9300	0.138	.7593	0.3516	0.369
.7966	0.5039	0.093	.7262	0.7093	0.394	.6637	0.9333	0.121	.7607	0.3549	0.377
.7980	0.5072	0.085	.7281	0.7137	0.406	.6651	0.9366	0.109	.7620	0.3579	0.367
.7994	0.5105	0.085	.7295	0.7170	0.403	.6665	0.9399	0.100	.7635	0.3615	0.367
.8047	0.5232	0.080	.7354	0.7310	0.411	.6680	0.9434	0.080	7140.6306	0.5357	−0.102
.8061	0.5265	0.070	.7369	0.7345	0.410	.6732	0.9558	0.015	.6320	0.5390	0.100
.8076	0.5300	0.080	.7381	0.7376	0.404	.6753	0.9607	0.016	.6335	0.5425	0.115
.8089	0.5330	0.074	.7396	0.7411	0.410	.6769	0.9646	+0.004	.6419	0.5623	0.170
.8149	0.5473	0.109	.7453	0.7546	0.425	.6783	0.9679	0.006	.6434	0.5659	0.186
.8176	0.5536	0.134	.7468	0.7582	0.412	.6797	0.9712	0.014	.6448	0.5692	0.196
.8251	0.5715	0.193	.7482	0.7614	0.433	.6811	0.9745	0.031	.6468	0.5741	0.198
.8272	0.5764	0.214	.7495	0.7645	0.423	.6866	0.9877	0.042	.6519	0.5862	0.213
3.6040	0.4193	−0.193	.7612	0.7922	0.393	.6880	0.9909	0.044	.6536	0.5900	0.247
.6054	0.4226	0.211	.7626	0.7955	0.390	.6895	0.9945	0.044	.6550	0.5933	0.252
.6068	0.4259	0.209	.7640	0.7988	0.389	.6908	0.9975	0.042	.6563	0.5966	0.264
.6083	0.4295	0.186	.7653	0.8021	0.396	.6967	0.0115	0.023	.6618	0.6095	0.300
.6145	0.4443	0.137	.7714	0.8164	0.382	.6989	0.0168	0.031	.6633	0.6131	0.320
.6158	0.4473	0.142	.7728	0.8197	0.385	.7019	0.0239	0.020	.6647	0.6164	0.326
.6172	0.4506	0.136	.7741	0.8229	0.376	7132.6713	0.6535	−0.357	.6661	0.6197	0.339
.6186	0.4539	0.123	.7755	0.8262	0.367	.6728	0.6571	0.365	.6674	0.6230	0.344
.6248	0.4687	0.073	.7815	0.8405	0.338	.6741	0.6601	0.365	.6779	0.6477	0.359
.6266	0.4729	0.065	.7832	0.8444	0.333	.6757	0.6640	0.359	.6792	0.6510	0.369
.6281	0.4764	0.075	.7846	0.8477	0.330	.6817	0.6782	0.363	.6806	0.6543	0.365
.6302	0.4814	0.074	.7860	0.8510	0.328	.6831	0.6815	0.380	.6819	0.6573	0.370
.6360	0.4951	0.087	.7914	0.8639	0.313	.6845	0.6848	0.383	.6883	0.6724	0.388
.6373	0.4984	0.072	.7929	0.8674	0.297	.6860	0.6884	0.388	.6897	0.6757	0.391
.6387	0.5017	0.075	.7943	0.8707	0.299	.6915	0.7013	0.400	.6911	0.6790	0.388
.6401	0.5050	0.070	.7957	0.8740	0.288	.6929	0.7046	0.409	.6924	0.6823	0.388
.6461	0.5193	0.083	.8012	0.8872	0.254	.6945	0.7084	0.418	.6980	0.6955	0.402
.6475	0.5226	0.081	.8026	0.8905	0.256	.6959	0.7117	0.411	.6994	0.6988	0.404
.6489	0.5259	0.086	.8041	0.8941	0.237	.7032	0.7290	0.399	.7008	0.7021	0.403
.6502	0.5289	0.095	.8054	0.8971	0.243	.7047	0.7326	0.413	.7022	0.7053	0.403
.6557	0.5421	0.113	.8135	0.9163	0.169	.7062	0.7362	0.408	.7077	0.7185	0.413
.6572	0.5456	0.124	.8149	0.9196	0.160	.7076	0.7395	0.411	.7091	0.7218	0.415
.6586	0.5489	0.133	.8163	0.9229	0.150	.7088	0.7425	0.397	.7105	0.7251	0.409
.6600	0.5522	0.141	.8177	0.9262	0.146	7136.6982	0.2066	−0.405	.7168	0.7400	0.415
.6650	0.5640	0.168	7114.6034	0.7902	−0.383	.6996	0.2099	0.412	.7181	0.7433	0.417
.6665	0.5676	0.181	.6049	0.7938	0.388	.7010	0.2132	0.418	.7195	0.7466	0.418
.6679	0.5709	0.183	.6064	0.7974	0.396	.7024	0.2165	0.420	.7208	0.7496	0.422
.6693	0.5742	0.203	.6079	0.8009	0.394	.7079	0.2297	0.415	.7264	0.7627	0.413
.6753	0.5885	0.237	.6189	0.8270	0.355	.7093	0.2330	0.419	.7279	0.7663	0.403
.6769	0.5923	0.248	.6204	0.8306	0.358	.7114	0.2379	0.426	.7292	0.7696	0.409
.6783	0.5956	0.246	.6217	0.8336	0.363	.7170	0.2511	0.419	.7306	0.7729	0.405
.6799	0.5994	0.260	.6232	0.8372	0.360	.7197	0.2577	0.423	.7362	0.7861	0.384
.6913	0.6263	0.326	.6304	0.8542	0.330	.7211	0.2610	0.405	.7380	0.7905	0.395
.6927	0.6296	0.336	.6318	0.8575	0.301	.7281	0.2775	0.424	.7399	0.7949	0.382
.6940	0.6329	0.336	.6331	0.8608	0.319	.7295	0.2808	0.411	.7413	0.7982	0.382

TABLE IIIb. Blue observations of AK Herculis.

Hel. 000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm
7.6058	0.8711	−0.290	6757.6252	0.9172	−0.177	6757.6432	0.9598	0.000	6757.6627	0.0062	+0.056
.6072	0.8743	0.285	.6265	0.9202	0.158	.6447	0.9633	+0.018	.6641	0.0094	0.049
.6086	0.8776	0.277	.6321	0.9334	0.090	.6530	0.9831	0.046	.6654	0.0125	0.071
.6098	0.8807	0.265	.6340	0.9380	0.082	.6543	0.9861	0.059	.6672	0.0169	0.052
.6224	0.9106	0.200	.6361	0.9430	0.066	.6558	0.9897	0.050	.6745	0.0342	0.028
.6237	0.9136	0.182	.6376	0.9466	0.046	.6572	0.9930	0.059	.6761	0.0380	0.006

TABLE IIIb. (Continued).

JD Hel. 2430000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm
6757.6775	0.0413	−0.006	7102.6621	0.4614	−0.078	7111.6946	0.8895	−0.226	7112.6401	0.1327	−0.3
.6789	0.0446	0.019	.6635	0.4647	0.088	.7008	0.9043	0.199	.6414	0.1357	0.3
.6842	0.0572	0.072	.6648	0.4677	0.070	.7022	0.9076	0.193	.6471	0.1492	0.3
.6856	0.0605	0.084	.6709	0.4823	0.074	.7036	0.9109	0.182	.6485	0.1525	0.3
.6869	0.0636	0.110	.6724	0.4859	0.065	.7050	0.9142	0.168	.6497	0.1555	0.3
.6884	0.0671	0.124	.6738	0.4892	0.089	.7110	0.9285	0.127	.6512	0.1591	0.3
.6898	0.0704	0.127	.6792	0.5021	0.048	.7124	0.9318	0.117	.6566	0.1717	0.3
.6954	0.0836	0.168	.6805	0.5051	0.039	.7138	0.9351	0.101	.6580	0.1750	0.3
.6967	0.0869	0.188	.6820	0.5087	0.058	.7152	0.9384	0.081	.6593	0.1783	0.3
.6981	0.0902	0.191	.6888	0.5248	0.075	.7207	0.9515	0.039	.6606	0.1813	0.3
.7002	0.0951	0.208	.6902	0.5281	0.088	.7221	0.9548	0.011	.6716	0.2074	0.4
.7015	0.0981	0.215	.6917	0.5317	0.076	.7235	0.9581	0.011	.6730	0.2107	0.4
.7072	0.1117	0.257	.6973	0.5449	0.098	.7249	0.9614	+0.008	.6744	0.2140	0.4
.7088	0.1154	0.279	.6986	0.5479	0.096	.7263	0.9647	0.018	.6757	0.2170	0.4
.7103	0.1190	0.298	.7001	0.5514	0.129	.7277	0.9680	0.024	.6818	0.2316	0.4
.7120	0.1231	0.304	.7056	0.5647	0.186	.7326	0.9798	0.060	.6834	0.2354	0.4
.7176	0.1363	0.312	.7069	0.5677	0.188	.7339	0.9829	0.066	.6848	0.2387	0.4
.7190	0.1396	0.320	.7083	0.5710	0.208	.7353	0.9862	0.066	.6862	0.2420	0.4
.7204	0.1429	0.327	.7138	0.5842	0.233	.7367	0.9895	0.055	.6915	0.2546	0.4
.7217	0.1462	0.353	.7153	0.5877	0.229	.7381	0.9927	0.055	.6929	0.2579	0.4
.7273	0.1594	0.378	.7167	0.5910	0.245	.7394	0.9958	0.066	.6943	0.2612	0.4
.7287	0.1627	0.362	.7230	0.6058	0.249	.7409	0.9993	0.063	.6957	0.2645	0.4
.7304	0.1668	0.374	.7244	0.6091	0.280	.7457	0.0108	0.062	.7007	0.2763	0.4
7058.6432	0.0337	+0.026	.7258	0.6124	0.305	.7471	0.0141	0.059	.7022	0.2799	0.4
.6446	0.0370	0.034	.7313	0.6256	0.347	.7485	0.0174	0.059	.7034	0.2829	0.4
.6461	0.0405	0.003	.7327	0.6289	0.337	.7499	0.0207	0.054	.7050	0.2865	0.4
.6521	0.0548	−0.055	.7341	0.6322	0.344	.7512	0.0237	0.071	.7158	0.3123	0.3
.6536	0.0584	0.060	7104.6254	0.1190	−0.277	.7527	0.0273	0.046	.7172	0.3156	0.4
.6550	0.0617	0.076	.6278	0.1248	0.283	.7573	0.0383	0.023	.7186	0.3189	0.3
.6651	0.0856	0.164	.6371	0.1467	0.336	.7587	0.0416	0.003	.7200	0.3222	0.3
.6666	0.0892	0.174	.6388	0.1509	0.337	.7603	0.0455	−0.017	.7256	0.3354	0.3
.6682	0.0930	0.177	.6416	0.1574	0.329	.7617	0.0487	0.039	.7271	0.3389	0.3
.6736	0.1059	0.233	.6517	0.1813	0.382	.7631	0.0520	0.048	.7283	0.3419	0.3
.6751	0.1095	0.238	.6530	0.1846	0.364	.7694	0.0671	0.100	.7297	0.3452	0.3
.6765	0.1128	0.249	.6681	0.2203	0.396	.7707	0.0702	0.112	.7352	0.3582	0.3
.6890	0.1424	0.309	.6699	0.2247	0.396	.7721	0.0734	0.125	.7367	0.3617	0.3
.6904	0.1457	0.323	.6717	0.2288	0.405	.7735	0.0767	0.148	.7381	0.3650	0.3
.6955	0.1578	0.332	.6792	0.2467	0.426	.7800	0.0921	0.197	.7394	0.3683	0.3
.6970	0.1614	0.345	.6809	0.2508	0.423	.7822	0.0973	0.219	.7457	0.3831	0.3
.6984	0.1647	0.341	.6823	0.2541	0.411	.7835	0.1004	0.220	.7478	0.3881	0.3
.7071	0.1853	0.382	.6886	0.2689	0.418	.7848	0.1037	0.225	.7490	0.3911	0.3
.7085	0.1886	0.393	.6900	0.2722	0.418	.7909	0.1179	0.274	.7506	0.3947	0.2
.7099	0.1919	0.385	.6919	0.2769	0.396	.7929	0.1229	0.287	.7519	0.3980	0.2
.7147	0.2034	0.392	.6978	0.2909	0.385	.7957	0.1295	0.299	.7577	0.4117	0.2
.7161	0.2067	0.388	.7000	0.2961	0.400	.8030	0.1468	0.338	.7590	0.4147	0.2
.7176	0.2102	0.400	.7019	0.3005	0.397	.8047	0.1509	0.343	.7605	0.4183	0.2
.7240	0.2253	0.410	.7034	0.3041	0.387	.8061	0.1542	0.342	.7658	0.4309	0.1
.7254	0.2286	0.417	.7147	0.3310	0.362	.8076	0.1577	0.356	.7672	0.4342	0.1
.7268	0.2319	0.416	.7162	0.3345	0.376	.8134	0.1715	0.368	.7686	0.4375	0.1
.7369	0.2561	0.404	.7176	0.3378	0.352	.8147	0.1745	0.379	.7703	0.4416	0.1
.7383	0.2594	0.409	.7243	0.3538	0.356	.8161	0.1778	0.380	.7762	0.4556	0.1
.7397	0.2627	0.410	.7257	0.3571	0.337	.8175	0.1811	0.386	.7776	0.4589	0.0
.7452	0.2756	0.407	.7271	0.3604	0.343	7112.5929	0.0207	+0.057	.7790	0.4622	0.0
.7467	0.2792	0.402	.7508	0.4166	0.215	.5943	0.0240	0.049	.7804	0.4655	0.0
.7480	0.2825	0.402	.7524	0.4202	0.199	.5962	0.0284	0.060	.7859	0.4784	0.0
.7534	0.2951	0.394	.7541	0.4243	0.194	.5975	0.0317	0.063	.7874	0.4820	0.0
.7549	0.2987	0.395	7111.6042	0.6750	−0.374	.5989	0.0350	0.042	.7887	0.4853	0.0
.7564	0.3022	0.385	.6060	0.6794	0.381	.6002	0.0380	0.032	.7901	0.4886	0.0
7102.6098	0.3373	−0.387	.6078	0.6836	0.390	.6016	0.0413	0.019	.7958	0.5020	0.0
.6112	0.3406	0.388	.6196	0.7116	0.417	.6072	0.0545	−0.033	.7973	0.5056	0.0
.6126	0.3439	0.371	.6212	0.7154	0.415	.6085	0.0578	0.048	.7987	0.5089	0.0
.6244	0.3719	0.346	.6226	0.7187	0.404	.6106	0.0627	0.067	.8001	0.5122	0.0
.6258	0.3752	0.338	.6284	0.7324	0.399	.6121	0.0663	0.082	.8054	0.5248	0.0
.6271	0.3785	0.336	.6297	0.7357	0.409	.6178	0.0797	0.154	.8069	0.5284	0.0
.6333	0.3930	0.273	.6312	0.7393	0.405	.6192	0.0830	0.179	.8082	0.5314	0.0
.6347	0.3963	0.273	.6325	0.7423	0.411	.6208	0.0869	0.195	.8097	0.5350	0.0
.6362	0.3999	0.253	.6390	0.7577	0.396	.6230	0.0921	0.196	.8157	0.5492	0.0
.6452	0.4213	0.208	.6404	0.7610	0.409	.6280	0.1039	0.230	.8170	0.5523	0.1
.6465	0.4243	0.211	.6418	0.7643	0.398	.6293	0.1069	0.240	.8184	0.5556	0.1
.6479	0.4276	0.186	.6432	0.7676	0.396	.6305	0.1099	0.242	.8259	0.5734	0.1
.6535	0.4411	0.155	.6839	0.8642	0.296	.6320	0.1135	0.257	.8279	0.5781	0.1
.6548	0.4441	0.133	.6896	0.8777	0.259	.6374	0.1261	0.306	7113.6048	0.4212	−0.1
.6562	0.4474	0.110	.6917	0.8826	0.260	.6387	0.1294	0.316	.6061	0.4243	0.2

TABLE IIIb. (Continued).

JD Hel. 430000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm	JD Hel. 2430000+	Phase	Δm
113.6076	0.4278	−0.197	7113.7461	0.7565	−0.403	7114.6644	0.9349	−0.099	7136.7427	0.3121	−0.419
.6090	0.4311	0.175	.7475	0.7598	0.408	.6659	0.9385	0.090	.7440	0.3154	0.408
.6151	0.4457	0.128	.7489	0.7631	0.405	.6673	0.9418	0.059	.7510	0.3318	0.390
.6165	0.4490	0.133	.7503	0.7664	0.398	.6686	0.9448	0.044	.7530	0.3365	0.378
.6179	0.4523	0.105	.7619	0.7939	0.382	.6743	0.9585	+0.009	.7545	0.3401	0.392
.6193	0.4556	0.099	.7633	0.7971	0.375	.6760	0.9624	0.020	.7600	0.3532	0.373
.6255	0.4704	0.051	.7645	0.8002	0.384	.6776	0.9662	0.023	.7613	0.3563	0.373
.6274	0.4748	0.051	.7660	0.8037	0.381	.6790	0.9695	0.040	.7628	0.3598	0.371
.6295	0.4797	0.071	.7720	0.8177	0.364	.6804	0.9728	0.061	.7642	0.3631	0.371
.6309	0.4830	0.051	.7734	0.8213	0.359	.6818	0.9761	0.061	7140.6313	0.5373	−0.089
.6366	0.4968	0.073	.7748	0.8246	0.345	.6873	0.9893	0.067	.6328	0.5409	0.104
.6380	0.5000	0.070	.7762	0.8279	0.337	.6887	0.9926	0.067	.6341	0.5439	0.110
.6394	0.5033	0.062	.7825	0.8427	0.327	.6902	0.9961	0.056	.6427	0.5642	0.165
.6408	0.5066	0.067	.7839	0.8460	0.322	.6915	0.9992	0.046	.6441	0.5675	0.179
.6468	0.5209	0.070	.7852	0.8493	0.325	.6982	0.0151	0.048	.6462	0.5725	0.199
.6482	0.5242	0.073	.7866	0.8526	0.310	.6996	0.0184	0.067	.6475	0.5757	0.199
.6496	0.5275	0.084	.7922	0.8658	0.276	.7012	0.0222	0.049	.6529	0.5884	0.246
.6510	0.5308	0.079	.7936	0.8691	0.271	.7026	0.0255	0.041	.6542	0.5917	0.262
.6565	0.5440	0.107	.7950	0.8724	0.280	7132.6720	0.6552	−0.357	.6556	0.5950	0.257
.6579	0.5473	0.138	.7964	0.8756	0.279	.6734	0.6584	0.367	.6570	0.5983	0.270
.6592	0.5503	0.117	.8018	0.8886	0.234	.6748	0.6617	0.351	.6626	0.6115	0.306
.6606	0.5536	0.133	.8034	0.8924	0.229	.6764	0.6656	0.358	.6640	0.6147	0.314
.6659	0.5662	0.182	.8048	0.8957	0.232	.6825	0.6799	0.365	.6654	0.6180	0.318
.6672	0.5692	0.184	.8061	0.8987	0.230	.6838	0.6832	0.374	.6668	0.6213	0.316
.6686	0.5725	0.196	.8142	0.9180	0.147	.6852	0.6864	0.381	.6680	0.6244	0.320
.6700	0.5758	0.201	.8156	0.9213	0.137	.6866	0.6897	0.381	.6786	0.6493	0.354
.6761	0.5904	0.227	.8170	0.9245	0.119	.6922	0.7029	0.414	.6800	0.6526	0.364
.6776	0.5939	0.243	.8184	0.9278	0.106	.6937	0.7065	0.412	.6813	0.6559	0.368
.6791	0.5975	0.243	7114.6042	0.7921	−0.377	.6952	0.7101	0.417	.6827	0.6592	0.366
.6806	0.6011	0.259	.6057	0.7957	0.386	.6966	0.7133	0.419	.6890	0.6741	0.384
.6918	0.6277	0.316	.6071	0.7990	0.385	.7068	0.7375	0.422	.6902	0.6771	0.371
.6934	0.6313	0.318	.6086	0.8026	0.397	.7083	0.7411	0.414	.6918	0.6807	0.383
.6946	0.6343	0.321	.6197	0.8289	0.357	.7095	0.7441	0.419	.6931	0.6839	0.383
.6962	0.6382	0.329	.6211	0.8322	0.359	7136.6988	0.2080	−0.420	.6987	0.6971	0.402
.7046	0.6579	0.349	.6225	0.8355	0.352	.7003	0.2116	0.422	.7000	0.7001	0.400
.7060	0.6612	0.343	.6239	0.8388	0.347	.7016	0.2146	0.432	.7015	0.7037	0.408
.7074	0.6645	0.345	.6297	0.8525	0.317	.7031	0.2182	0.425	.7029	0.7070	0.388
.7087	0.6678	0.347	.6311	0.8558	0.314	.7086	0.2314	0.437	.7084	0.7202	0.417
.7140	0.6802	0.372	.6325	0.8591	0.302	.7106	0.2360	0.428	.7098	0.7235	0.424
.7156	0.6840	0.384	.6338	0.8624	0.305	.7120	0.2393	0.439	.7111	0.7265	0.412
.7172	0.6879	0.382	.6394	0.8756	0.275	.7177	0.2528	0.426	.7160	0.7380	0.418
.7186	0.6912	0.384	.6408	0.8789	0.269	.7190	0.2560	0.436	.7174	0.7416	0.420
.7241	0.7043	0.389	.6422	0.8822	0.265	.7204	0.2593	0.439	.7187	0.7446	0.420
.7255	0.7076	0.387	.6436	0.8855	0.260	.7218	0.2626	0.431	.7202	0.7482	0.418
.7273	0.7117	0.382	.6493	0.8992	0.222	.7288	0.2791	0.433	.7215	0.7512	0.409
.7288	0.7153	0.386	.6507	0.9025	0.203	.7302	0.2824	0.426	.7272	0.7647	0.398
.7300	0.7183	0.379	.6520	0.9055	0.198	.7316	0.2857	0.428	.7298	0.7710	0.393
.7362	0.7329	0.407	.6534	0.9088	0.176	.7331	0.2893	0.419	.7372	0.7885	0.383
.7377	0.7364	0.410	.6595	0.9234	0.135	.7399	0.3055	0.419	.7387	0.7921	0.388
.7390	0.7395	0.406	.6616	0.9283	0.135	.7413	0.3088	0.408	.7427	0.8015	0.371
.7403	0.7428	0.412	.6630	0.9316	0.113						

lished by Seyfert and Mason is geocentric, a light-time correction was applied to make the time heliocentric. In the data published by Labs and Stock the formulae and residuals are not in agreement. The correct epoch was recomputed from the detailed data kindly sent to me by the last author. Woodward's ephemeris was used to compute the *O–C*'s:

$$\text{JD Hel. Min.I} = 2422977.254 + 0\overset{\mathrm{d}}{.}42152207\,E.$$

Figure 1 illustrates the changes in period which have occurred. Crosses refer to Harvard patrol plates, small dots are means of visual data, large dots are photographic data, squares are photoelectric observations. The open symbols and dotted curve refer to secondary minima.

The period used in the final reduction of the present observations is:

$$\text{JD Hel. Min.I} = 2436757.6601 + 0\overset{\mathrm{d}}{.}42152502\,E.$$

Tables IIIa and IIIb give the heliocentric Julian Day, the phase counted from primary minimum, and the magnitude difference between the variable star and the comparison star. The observations of four successive nights of exceptionally good quality in June 1960 showed definitely that there was an interval of constant light during the minima of 35 minutes' duration. Tables IVa and IVb and Fig. 2 show the normal points of all eleven nights. A zero point correction of +0.419 magnitude for yellow and +0.425 magnitude for blue light was applied to make the magnitude difference equal to zero

Table IVa. Normal points of yellow observations.

Phase	Obs. Δm	n	Corr. Δm	O–C	Rect. Δm
0.0116	+0.447	9	+0.461	−0.001	+0.190
0.0245	0.441	9	0.455	+0.004	+0.190
0.0364	0.421	9	0.434	+0.014	+0.179
0.0481	0.378	8	0.389	+0.004	+0.151
0.0605	0.321	8	0.330	−0.003	+0.112
0.0740	0.266	8	0.274	−0.002	+0.078
0.0876	0.217	8	0.223	−0.004	+0.050
0.1007	0.177	9	0.182	+0.001	+0.030
0.1147	0.133	10	0.137	−0.005	+0.008
0.1323	0.101	12	0.104	+0.001	+0.001
0.1480	0.079	10	0.081	+0.001	0.000
0.1625	0.059	13	0.061	−0.001	−0.001
0.1803	0.043	11	0.044	+0.003	+0.003
0.2087	0.006	10	0.006	−0.009	−0.009
0.2258	0.002	10	0.002	−0.003	−0.003
0.2440	0.000	10	0.000	0.000	0.000
0.2615	0.004	10	0.004	+0.005	+0.005
0.2795	0.003	10	0.003	0.000	0.000
0.2982	0.019	10	0.019	+0.007	+0.007
0.3238	0.028	11	0.029	−0.003	−0.003
0.3416	0.051	10	0.052	+0.003	+0.003
0.3606	0.064	10	0.066	−0.005	−0.005
0.3831	0.115	6	0.118	+0.008	+0.017
0.4009	0.154	6	0.158	+0.004	+0.034
0.4184	0.205	6	0.211	+0.004	+0.062
0.4281	0.227	8	0.234	0.000	+0.073
0.4454	0.273	8	0.281	−0.009	+0.099
0.4606	0.317	7	0.326	−0.001	+0.129
0.4774	0.349	8	0.359	+0.013	+0.150
0.4943	0.342	8	0.352	+0.001	+0.140
0.5081	0.337	7	0.347	−0.004	+0.135
0.5272	0.338	7	0.348	+0.005	+0.141
0.5427	0.305	8	0.314	0.000	+0.119
0.5575	0.261	8	0.268	−0.005	+0.089
0.5704	0.224	8	0.230	−0.004	+0.066
0.5865	0.179	8	0.184	−0.002	+0.040
0.6021	0.143	8	0.147	+0.003	+0.023
0.6224	0.086	8	0.088	−0.012	−0.008
0.6453	0.070	8	0.072	+0.002	+0.002
0.6648	0.052	10	0.053	+0.004	+0.004
0.6811	0.039	10	0.040	+0.006	+0.006
0.6994	0.022	10	0.023	+0.001	+0.001
0.7141	0.013	10	0.013	−0.001	−0.001
0.7329	0.012	10	0.012	+0.003	+0.003
0.7454	0.006	10	0.006	−0.003	−0.002
0.7644	0.006	10	0.006	−0.006	−0.006
0.7938	0.031	10	0.032	+0.002	+0.002
0.8217	0.045	10	0.046	−0.009	−0.008
0.8523	0.094	8	0.096	+0.003	+0.003
0.8721	0.123	10	0.126	+0.004	+0.005
0.8866	0.157	10	0.161	+0.007	+0.018
0.9062	0.203	10	0.209	−0.001	+0.035
0.9224	0.258	10	0.265	−0.001	+0.066
0.9348	0.304	8	0.313	−0.001	+0.094
0.9502	0.364	8	0.375	−0.002	+0.134
0.9645	0.414	9	0.427	−0.004	+0.168
0.9826	0.449	7	0.463	+0.004	+0.191
0.9931	0.448	8	0.462	−0.003	+0.189

Table IVb. Normal points of blue observations.

Phase	Obs. Δm	n	Corr. Δm	O–C	Rect. Δm
0.0134	+0.483	9	+0.498	−0.003	+0.18
0.0258	0.477	10	0.492	+0.004	+0.18
0.0381	0.445	9	0.459	+0.004	+0.16
0.0499	0.390	8	0.402	−0.004	+0.13
0.0623	0.344	8	0.354	0.000	+0.10
0.0755	0.286	8	0.294	0.000	+0.07
0.0895	0.240	8	0.247	+0.005	+0.05
0.1023	0.200	9	0.205	+0.007	+0.03
0.1165	0.153	10	0.157	0.000	+0.01
0.1339	0.116	10	0.119	+0.003	+0.00
0.1500	0.088	10	0.090	0.000	0.00
0.1636	0.068	12	0.070	−0.001	−0.00
0.1825	0.045	10	0.046	−0.001	−0.00
0.2104	0.011	10	0.011	−0.009	−0.00
0.2277	0.009	10	0.009	0.000	0.00
0.2470	0.000	10	0.000	−0.003	−0.00
0.2637	0.010	11	0.010	+0.008	+0.00
0.2811	0.011	10	0.011	+0.004	+0.00
0.3003	0.024	11	0.025	+0.008	+0.00
0.3254	0.039	10	0.040	+0.001	+0.00
0.3433	0.053	10	0.054	−0.004	−0.00
0.3622	0.073	10	0.075	−0.007	−0.00
0.3848	0.112	6	0.115	−0.009	+0.00
0.4025	0.167	6	0.172	+0.003	+0.03
0.4203	0.220	6	0.226	+0.004	+0.05
0.4297	0.241	8	0.248	−0.005	+0.06
0.4471	0.301	8	0.310	+0.002	+0.10
0.4623	0.340	7	0.350	+0.003	+0.12
0.4795	0.363	8	0.374	+0.006	+0.141
0.4960	0.358	8	0.369	−0.003	+0.13
0.5097	0.359	7	0.370	−0.002	+0.13
0.5287	0.354	10	0.365	+0.006	+0.13
0.5444	0.321	8	0.330	+0.004	+0.11
0.5584	0.277	9	0.285	−0.002	+0.08
0.5722	0.231	8	0.237	−0.004	+0.05
0.5882	0.190	8	0.195	+0.003	+0.03
0.6038	0.154	8	0.158	+0.006	+0.021
0.6240	0.102	8	0.105	−0.005	−0.00
0.6469	0.077	8	0.079	−0.001	−0.001
0.6665	0.064	10	0.066	+0.008	+0.00
0.6829	0.046	10	0.047	+0.004	+0.00
0.7010	0.029	10	0.030	0.000	0.00
0.7158	0.019	10	0.019	−0.003	−0.00
0.7354	0.015	9	0.015	−0.003	−0.00
0.7470	0.013	10	0.013	−0.005	−0.00
0.7647	0.024	8	0.025	+0.002	+0.001
0.7956	0.044	9	0.045	+0.002	+0.001
0.8233	0.065	10	0.067	−0.006	−0.00
0.8538	0.112	9	0.115	0.000	0.00
0.8738	0.149	10	0.153	+0.003	+0.00
0.8884	0.180	9	0.185	+0.003	+0.01
0.9077	0.233	10	0.240	−0.001	+0.03
0.9241	0.290	10	0.298	−0.003	+0.06
0.9366	0.334	8	0.344	−0.009	+0.08
0.9520	0.400	8	0.412	−0.006	+0.131
0.9660	0.451	8	0.465	−0.004	+0.164
0.9841	0.485	8	0.501	−0.001	+0.185
0.9948	0.483	8	0.498	−0.007	+0.182

at maximum light. The probable error of a normal point consisting of about nine observations was found to be ±0.0015 magnitude for yellow and ±0.0018 magnitude for blue light.

4. ORBITAL ELEMENTS

The magnitudes and phases of the normal points were converted into intensities and phase angles. The intensities were corrected to remove the effect of the light of the faint component, which was taken to be four magnitudes fainter than the light of the variable at maximum. Least-squares solutions were made for the light in the maxima according to the formula:

$$l=A_0+A_1\cos\theta+A_2\cos2\theta+B_1\sin\theta+B_2\sin2\theta.$$

The values of the constants are given in Table V together with their probable errors. The coefficient of the $\cos\theta$ term has the sign predicted by theory.

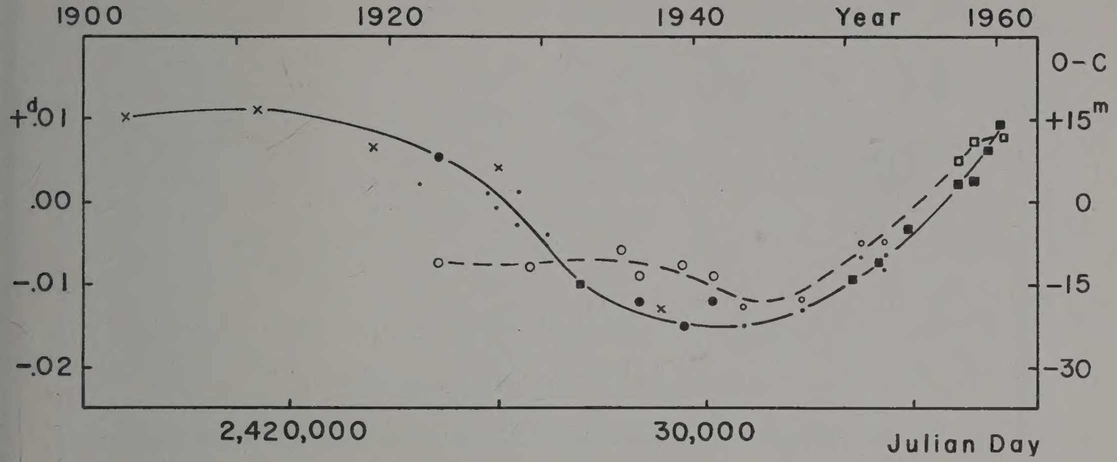

FIG. 1. Period change of AK Herculis. The crosses refer to Harvard patrol plates, small dots represent mean values of visual obser- ations, large dots denote photographic data, squares represent photoelectric observations. The open symbols and the dotted curve efer to secondary minima.

The intensities were first rectified for the small sinθ nd sin2θ terms, which can be called perturbations. The ifference between the reflection coefficients was found om the observed quantity A_1 and the ratio of the coef- cients was found from the ratio of the depths of inima after a provisional rectification had been made. hen the usual rectifications for reflection and oblate- ess were carried out separately, and the phase angles ere rectified. After this final rectification the minima were slightly displaced from the expected phase, namely $+\frac{1}{2}°$ for the primary and $-1°$ for the secondary minimum.

The orbital solution shows that primary minimum is caused by annular eclipse and secondary minimum by the total eclipse. The curvature of the eclipse curve around the time of primary minimum is slight which indicates a small coefficient of limb darkening for the greater star. Using the ψ functions, theoretical light

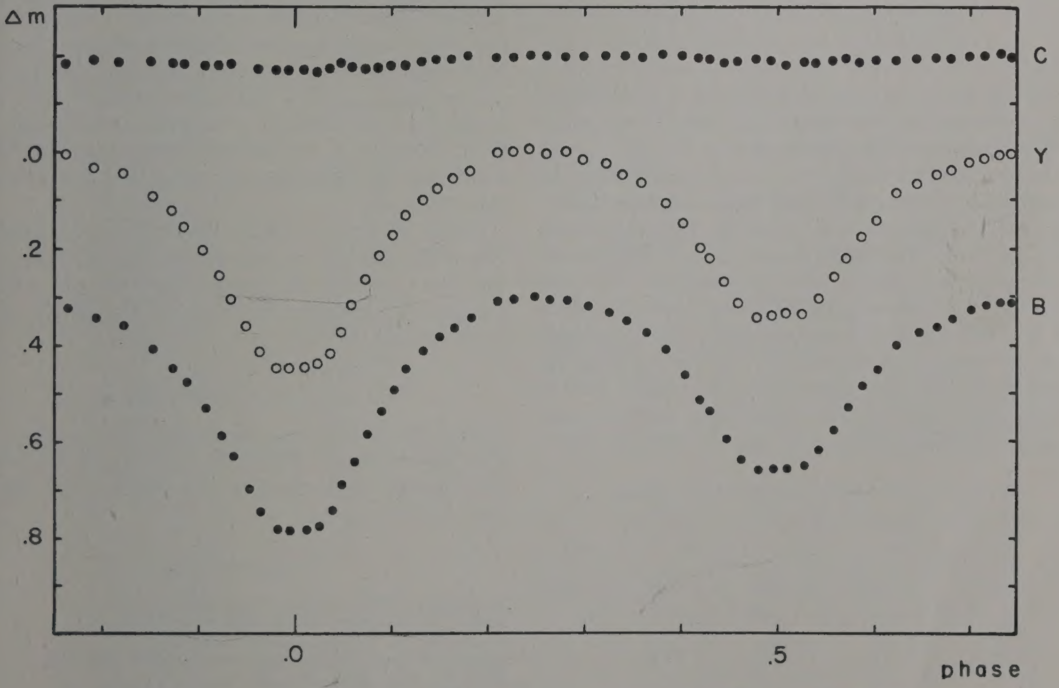

FIG. 2. The light curves and color curve of AK Herculis. The observations are B=blue, Y=yellow, C=color.

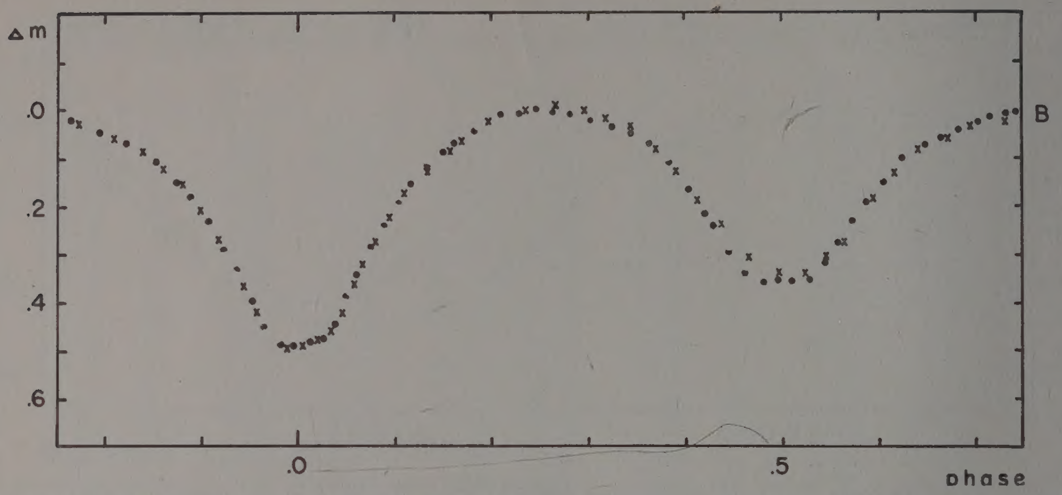

FIG. 3. Change in the blue light curve of AK Herculis as found from a comparison between the normals of Schmidt and Hercze (1957–58) and ours (1960). The dots refer to our observations.

curves were computed for $x=0.4$ and 0.6, and for $k=0.40$, 0.45, and 0.50. It was found that the following solution achieves the best agreement with the observations:

$x=0.4$	$k=0.45$
$\alpha_0^{oc}=1.00$	$\alpha_0^{tr}=1.03$
$a_g=0.54$	$a_s=0.24$
$b_g=0.47$	$b_s=0.21$
$\epsilon=0.13$	$i=77\overset{\circ}{.}6$
Yellow	*Blue*
$L_g=0.880$	$L_g=0{\cdot}883$
$L_s=0.120$	$L_s=0.117$
$I_g=1.49\,I_s$	$I_g=1.52\,I_s$

In this list $\epsilon=(a-b)/a$ is the oblateness, i is the inclination corrected for polar flattening, and I is the mean surface brightness of the components.

The observed normal points were compared with the theoretical ones computed from these elements. Tables IVa and IVb show the *O–C*'s together with the normal points of the rectified light curves. The probable error of one observed normal point as compared with the theoretical light curve is ±0.0034 magnitude for yellow and ±0.0030 magnitude for blue light. These probable errors are larger than those derived previously from the spread in the observed light curves. The model used in the computation of the theoretical light curves is only an approximation of the real star. At certain phases there are slight systematic deviations between th observed light curv which cannot be removed by choosing another mode

The most recent photoelectric observations are thos published by Schmidt and Herczeg. The plot of thei observations shows a systematic difference in the phas direction between the observations made in 1957 an those made in 1958. This is caused by the fact that the used Woodward's period for the reduction of thei observations and this period was not valid at the tim of these observations. New phases were computed usin our period and the systematic effect between the obser vations made in the two years vanished. In Fig. 3 ou normal points in blue light are compared with thos found from the material of Schmidt and Herczeg. Th main difference is the shift of the secondary minimum which has also been found to occur in other systems this type.

From his radial velocity curve Sanford found th values $K_1=78$ km/sec and $a_1 \sin i=452\,110$ km. Usin the latter we find the distance between the center the greater, more massive star and the gravity cent to be $a_1=463\,000$ km.

TABLE V. Coefficients of the light in the maxima.

λ	A_0	A_1	A_2	B_1	B_2
Y	+0.9078	−0.0164	−0.0885	+0.0048	+0.0015
	± 13	± 14	± 18	± 7	± 9
B	+0.8947	−0.0211	−0.0955	+0.0074	+0.0020
	± 12	± 13	± 18	± 7	± 8

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NOTES AND OBSERVATIONS

Times of Minimum Light for Several Eclipsing Binaries

Times of minima were determined from film-strip exposures for 14 eclipsing binaries, leading to three improved periods.

Through the kindness of Dr. E. F. Carpenter of the eward Observatory 35-mm film strips of several lipsing binaries were made available for instruction at t. Holyoke College. The films were reduced according the technique given by Koch (1956). For the most rt, times of minimum light were determined by secting chords of constant light drawn between the anches of the eclipse curves. The observed heliocentric mes are collected in Table I.

For several stars all available published minima were sted against the ephemerides given in 1960, *Rocznik Astron. Obs. Krakow* **31**. The list of references to the published data was kindly provided by Dr. F. B. Wood from the Card Catalog of Eclipsing Variables maintained at the University of Pennsylvania. From this study it was concluded that the given light elements for ZZ Cep and SS Lib are adequate and that the periods seem to be constant within the scatter of the residuals. It was further concluded that the ephemerides for several stars of constant period can be improved to the values listed in Table II.

TABLE I. Times of minimum light.

Star	JD (hel.) 2434000+
KP Aql	910.859
ZZ Cep	533.764
RZ Com	847.897
Z Dra	848.769
RZ Dra	888.71
SW Lac	910.854
SS Lib	868.71
V501 Oph	554.758
V508 Oph	861.877
TY Peg	554.928
AO Ser	859.806
Y Sex	838.74
RS Sct	850.879
AW Vir	886.711

TABLE II. Improved ephemerides.

Star	Epoch	Period
KP Aql	2426096.519	1.683732
V501 Oph	2430911.395	0.9679504
AO Ser	2434133.464	0.87934745

The students and assistants who participated in the exercise were K. Bracher, T. Campbell, J. Koch, N. Poland, D. Wethey, and S. Wyckoff.

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ROBERT H. KOCH
Amherst College Observatory,
Amherst, Massachusetts,
Received October 10, 1960.

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Transit of Mercury Observed at Austin, Texas

Third- and fourth-contact times of Mercury's November 7, 1960 transit were observed, with probable errors less than 1 sec.

The transit of Mercury of November 7, 1960 was observed at the student's observatory of the University of Texas in Austin. The observations were made by projection with the 9-inch refractor stopped down to 6 inches. The projected image of the sun was about 20 inches in diameter; the diameter of Mercury was about 0.1 inch. The half-second beats of a chronometer, checked to $\pm 0^{\rm s}.2$ against WWV before and after the transit, were recorded simultaneously with the observer's signals of several phases of the egress by means of a tape recorder. In addition to the direct observations of internal and external contacts, the distance between the limbs of the sun and of Mercury were estimated in tenths of the diameter of Mercury during the last three minutes of the transit. From these estimates, the time of contact could also be determined by interpolation. The observed and interpolated times of third and fourth contacts are as follows.

	Third contact	Fourth contact
Observed UT	$19^{\rm h}09^{\rm m}46^{\rm s}.0$	$19^{\rm h}11^{\rm m}47^{\rm s}.2$
Interpolated UT	19 09 48.0 ($n=15$)	19 11 46.7 ($n=$
Adopted UT	19 09 47.0±0.6	19 11 47.0±0.7

The probable errors are derived from the scatter o the $n=15$ or $n=9$ values of the interpolated times.

Geographic coordinates of the observatory:

Long. = $6^{\rm h}30^{\rm m}57^{\rm s}.7$ *W*,
Lat. = +30°16′28″,
Alt. = 210 m approx.

Observers and participants: Drs. D. H. Menzel, E. J Prouse, and G. de Vaucouleurs; H. D. Ables and P. R Jordahl.

G. DE VAUCOULEURS
Department of Astronomy,
University of Texas,
Received December 1, 1960

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Meridian Observations of the Moon, 1958–59

Latitude and longitude residuals from Ephemeris lunar positions are given for the lunations of 1958–59.

These observations, a continuation of those last published in *Astron. J.* **63**, 299, 1958, were made with the six-inch transit circle. The observers were J. W. Kitchens, R. W. Rhynsburger, W. T. Toland, D. K. Scott, and the authors. The details of reduction are given in *Astron. J.* **51**, 191, 1945. The residuals of the moon's longitude and latitude refer, as in the past, to the positions given in the *American Ephemeris*.

Corrections for limb irregularities according to Hayn were applied to the individual right ascension and declination residuals. In Table I the lunation means in ecliptic longitude and latitude are given, along with the annual means reduced to the mid-point of the year. The quantity H is the annual mean of the limb corrections used.

A. N. ADAMS AND S. M. BESTUL
U. S. Naval Observatory,
Washington 25, D. C.,
Received December 7, 1960.

TABLE I. Lunation and annual means.

1958				1959			
Lun.	δλ	δβ	n	Lun.	δλ	δβ	n
433	−4″.21	−0″.52	10	446	−5″.19	−0″.63	9
434	−4.32	−0.81	9	447	−4.60	−0.58	8
435	−4.19	−0.35	1	448	−4.82	−0.06	9
436	−3.86	−0.70	4	449	−4.60	−0.84	3
437	−3.04	−0.26	1	450	−4.32	−0.83	5
438	−4.62	−1.20	5	451	−4.92	−1.11	7
439	−4.41	−0.78	8	452	−4.87	−1.14	4
440	···	···	···	453	−5.07	−1.07	10
441	···	···	···	454	−4.96	−0.51	12
442	−4.63	−0.29	7	455	−5.28	−0.21	8
443	−4.54	+0.01	6	456	−5.14	−0.17	8
444	−4.98	−0.58	10	457	−5.08	−0.22	8
445	−4.54	−0.31	7				

Annual means

Year	δλ	H	δβ	H
1958	−4″.43	+0″.07	−0″.59	+0″.08
1959	−4.90	+0.01	−0.58	+0.09